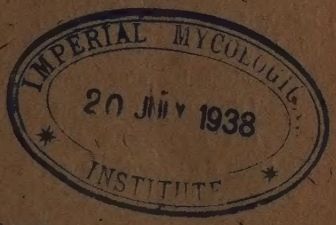
COMPTES RENDUS

(DOKLADY)

DE L'ACADÉMIE DES SCIENCES DE L'URSS

NOUVELLE SÉRIE

1937



VOLUME XVI

№ 1

ÉDITION DE L'ACADEMIE DES SCIENCES DE L'URSS
MOSCOU

AVIS AUX AUTEURS

Les «Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS» paraissent tous les dix jours, neuf numéros formant un volume d'environ 500 pages pourvu d'une table des matières.

Les «Comptes Rendus de l'Académie des Sciences de l'URSS» publient des communications courtes précisant l'essentiel des travaux plus volumineux en préparation. Ces articles, ne dépassant pas comme règle dix mille signes typographiques, sont donnés en l'une des trois langues: français, anglais, allemand. Ils embrassent les sciences mathématiques, physiques, naturelles et techniques, reproduisant les résultats de recherches scientifiques actuellement en cours ou à peine achevées qui représentent les plus récentes données de l'investigation scientifique.

Les auteurs ont droit chacun à 25 extraits gratuits.

Les manuscrits doivent être adressés à la Rédaction des «Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS» — Moscou, oulitza Koujbysheva, 8.

5 juillet

COMPTES RENDUS

(DOKLADY)

DE L'ACADÉMIE DES SCIENCES DE L'URSS

COMITÉ DE RÉDACTION

A. Borissiak, de l'Académie, **A. Fersman**, de l'Académie, **A.Frumkin**, de l'Académie, **A. Richter**, de l'Académie, **S. Wawilow**, de l'Académie

NOUVELLE SÉRIE

5-me année

Paraissant tous les dix jours

1937

VOLUME XVI, № 1

TABLE DES MATIÈRES

Pages

MATHÉMATIQUES

Kh. Smalickij. Sur les fonctions de Le Roi, H. Poincaré et V. Stekloff . 3
M. Keldyš (Keldysch) et **L. Sedov.** Sur la solution effective de quelques problèmes limites pour les fonctions harmoniques 7
Stefan Bergmann. Über einige Abschätzungen bei pseudokonformen Abbildungen 11

MÉCANIQUE

J. A. Mindlin. The Boundary Dynamic Problem of the Theory of Elasticity for a Circle with Given Displacements 15

THÉORIE DE L'ÉLASTICITÉ

V. Sokolovsvij. On the Design of a Spherical Shell 19

HYDROTECHNIQUE

V. S. Koslov. Determination of the Elements of Percolation Flow under Dams with three Cut-off Walls, Resting upon Permeable Foundation of Finite Depth. 25

PHYSIQUE MATHÉMATIQUE

V. D. Kupradze. Lösung der allgemeinen Aufgabe der Diffraktion von elektromagnetischen Wellen 31

PHYSIQUE

V. K. Arkadiev (W. Arkadiew), membre correspondant de l'Académie. La bande de dispersion et le skin-effet dans le champ sinusoïdal et dans celui de transition 35
A. Prokofjev. Torch Ion Counter 41

BIOPHYSIQUE

P. Lasareff, de l'Académie. Étude théorique de l'influence de l'ascension des montagnes et de l'influence des vents sur l'adaptation visuelle périphérique 45

GÉOPHYISIQUE

N. Mihal. Über die Bestimmung der Geoidfigur aus den Anomalien des Horizontalgradienten der Erdschwere 49

BIOCHIMIE

I. A. Smorodincev (Smorodinzew) et **A. M. Feldt.** Détermination de la constante de dissociation de la thyréoglobuline 51

GÉOCHIMIE

M. A. Kločko. The «Lake Age» of the Caspian Sea and Its Volume at the Time It Became a Closed Basin 55

SCIENCE DU SOL

K. S. Andrianov and **A. I. Smirnov.** On the Problem of the Genesis of Vivianite 59

GÉNÉTIQUE

P. O. Sitko. Frequency of Lethals in the *X*-Chromosome Due to the Irradiation of Spermatozoa in the Males and in the Spermathecae of the Females of *Drosophila melanogaster* 63

PHYSIOLOGIE VÉGÉTALE

I. N. Konovalov and **I. E. Rogalev.** The Behaviour of Nitrogenous Substances during the Yarovization of Plants 65

VIROLOGIE

V. L. Ryžkov and **A. M. Vovk.** A Neu Disease of the Onion (*Allium Cepa*) 69

Rédacteur en chef A. Fersman. — Rédacteur S. Bernstein

En composition: 11/VI 1937.—En impression: 15/VII 1937.—Nombre de feuilles d'imprimerie 4 1/2. Pages 1—72. Nombre de caractères par feuille imprimée 65.000.—Format du papier 72×110 cm. Tirage 1.300. Moscou. Glavlit B15321—Numéro d'édition 736. Numéro typographique 1023.

Imprimerie № 16. Moscou, Trekhproudnij peréoulok, 9.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

MATHÉMATIQUES

SUR LES FONCTIONS DE LE ROY, H. POINCARÉ ET V. STEKLOFF

Par Kh. SMALICKIJ

(Présenté par S. N. Bernstein, de l'Académie, le 13. V. 1937)

1. Considérons deux systèmes de potentiels de simple couche vérifiant sur une surface S de Liapounoff les conditions suivantes:

Le système I de Poincaré vérifie la condition:

$$\frac{dV_{ki}}{dn} = \alpha_k \frac{dV_{ke}}{dn}, \qquad k = 0,\ 1,\ 2,\ \ldots, \tag{1}$$

et le système II de Le Roy vérifie la condition:

$$\frac{dV_{ki}}{dn} - \frac{dV_{ke}}{dn} = \xi_k \varphi V_k, \text{ où } \varphi > 0; \qquad k = 0,\ 1,\ 2,\ \ldots \tag{2}$$

Ce système est complet sur la surface S.

Dans le cas d'une sphère ou bien d'un ellipsoïde les deux systèmes coïncident pour $\varphi = \rho$, où ρ est la densité de la couche électrostatique. Le Roy croyait que, si $\varphi = \rho$, les deux systèmes coïncident même pour une surface arbitraire. Nous démontrons dans cette note que cette hypothèse n'est pas exacte. Stekloff a remarqué le premier une inexactitude dans les raisonnements de Le Roy, mais il n'a pas réussi à démontrer qu'en général la coïncidence affirmée par Le Roy n'a pas lieu.

2. Le potentiel V_0 d'une simple couche, qui est constant dans le domaine intérieur à S est une fonction de Poincaré pour $\alpha_k = 0$. Si les deux systèmes coïncident, V_0 est aussi une fonction de Le Roy. Nous avons donc:

$$\frac{dV_{oi}}{dn} - \frac{dV_{oe}}{dn} = 4\pi\rho = \xi_0 \varphi V_0,$$

$V_0 = \text{const}$ sur S, d'où il suit que $\varphi = \frac{4\pi}{\xi_0 V_0}\rho$, c'est-à-dire, que φ est la densité de la couche électrostatique.

3. Considérons le système des fonctions de Le Roy en posant $\varphi = \rho$. Désignons par μ_k la densité du potentiel V_k. Comme on a:

$$\frac{dV_{ki}}{dn} - \frac{dV_{ke}}{dn} = 4\pi\mu_k = \xi_k \rho V_k,$$

$$4\pi \int_{(S_0)} \mu_k(0)\, \frac{1}{r_{10}}\, ds_0 = \xi_k \int_{(S_0)} \frac{\rho(0)\, V_k(0)}{r_{10}}\, ds_0 = 4\pi V_k(1), \tag{3}$$

où les chiffres entre parenthèses désignent des points de la surface S. Nous parvenons à une équation intégrale (3) pour les fonctions de Le Roy.

Remplaçons dans (1) $\frac{dV_{ki}}{dn}$ et $\frac{dV_{ke}}{dn}$ par leurs expressions au moyen de μ_k. On obtient:

$$2\pi(1+\alpha_k)\mu_k(1) = (1-\alpha_k)\int\limits_{(S_0)} \mu_k(0)\frac{\cos(r_{10}n_1)}{r_{10}^2}\,ds_0. \tag{4}$$

En appliquant la formule de Green et en remarquant que les égalités $\frac{dV_{ki}}{dn} - \frac{dV_{ke}}{dn} = 4\pi\mu_k$ et $\frac{dV_{ki}}{dn} = \alpha_k\frac{dV_{ke}}{dn}$ conduisent à $\frac{dV_{ki}}{dn} = \frac{4\pi\alpha_k}{\alpha_k - 1}\mu_k$, on obtient

$$2\pi(1+\alpha_k)V_k(1) = (1-\alpha_k)\int\limits_{(S_0)} V_k(0)\frac{\cos(r_{01}n_0)}{r_{01}^2}\,ds_0. \tag{5}$$

Or l'équation (1) donne

$$\int\limits_{(S)} V_k\frac{dV_{ki}}{dn}\,ds = \alpha_k\int\limits_{(S)} V_k\frac{dV_{ke}}{dn}\,ds = \int\limits_{(D_i)}(\nabla V_k)^2\,d\tau = -\alpha_k\int\limits_{(D_e)}(\nabla V_k)^2\,d\tau,$$

on en déduit $\alpha_k \leqslant 0$, $1-\alpha_k \neq 0$.

4. Pour les fonctions de Le Roy nous avons:

$$\mu_k = \frac{\xi_k}{4\pi}\rho V_k. \tag{6}$$

Si les fonctions de Le Roy coïncident avec les fonctions de H. Poincaré, les densités des potentiels coïncident aussi.

En substituant (6) dans (4) on obtient:

$$2\pi(1+\alpha_k)\rho(1)V_k(1) = (1-\alpha_k)\int\limits_{(S_0)}\rho(0)V_k(0)\frac{\cos(r_{10}n_1)}{r_{10}^2}\,ds_0. \tag{7}$$

En multipliant (7) par $\frac{1}{4\pi}\frac{\xi_k}{r_{12}}\,ds_1$, en étendant l'intégration à la surface S_1 et en utilisant (3) nous obtenons:

$$2\pi(1+\alpha_k)V_k(2) = \frac{\xi_k}{4\pi}(1-\alpha_k)\int\limits_{(S_0)}\rho(0)V_k(0)\Big(\int\limits_{(S_1)}\frac{1}{r_{12}}\frac{\cos(r_{10}n_1)}{r_{10}^2}\,ds_1\Big)ds_0. \tag{8}$$

De même l'équation (5) conduit à

$$2\pi(1+\alpha_k)V_k(2) = \frac{\xi_k}{4\pi}(1-\alpha_k)\int\limits_{(S_0)}V_k(0)\Big(\int\limits_{(S_1)}\frac{\rho(1)}{r_{12}}\frac{\cos(r_{01}n_1)}{r_{01}^2}\,ds_1\Big)ds_0. \tag{9}$$

La comparaison des équations (8) et (9) donne

$$\int\limits_{(S_0)}\rho(0)V_k(0)\Big[\int\limits_{(S_1)}\frac{1}{r_{12}}\Big(\frac{\cos(r_{10}n_1)}{r_{10}^2} - \frac{\rho(1)}{\rho(0)}\frac{\cos(r_{01}n_0)}{r_{01}^2}\Big)ds_1\Big]ds_0 = 0, \tag{10}$$

$$k = 0,\ 1,\ 2,\ \ldots$$

On en déduit que

$$\rho(0)\frac{\cos(r_{10}n_1)}{r_{10}^2} - \rho(1)\frac{\cos(r_{01}n_0)}{r_{01}^2} = 0, \tag{11}$$

car le système des fonctions V_k est complet et le noyau $\frac{1}{r_{12}}$ est fermé.

On peut donner à l'équation (11) une autre forme; on a en effet:

$$\frac{\cos(r_{10}n_1)}{r_{10}^2} = \rho(1)\,S(0,\ 1), \tag{12}$$

où S est une fonction symétrique des points (0) et (1).

L'équation (11) donne $\frac{\cos(r_{10}n_1)}{\cos(r_{01}n_0)} = \frac{\rho(1)}{\rho(0)} > 0$. Cette condition ne peut être satisfaite que pour les surfaces convexes.

5. La condition (11) n'est pas seulement nécessaire, mais elle est encore suffisante. Cela découle des propositions suivantes:

a) Si $\varphi(1)\,K_1(1, 2)$ et $\varphi(1)\,K_2(1, 2)$ sont les noyaux de Schmidt, le noyau $\varphi(1)\,K_1(1, 2)$ étant fermé, et si la fonction

$$\Phi(0, 2) = \int_{(S_1)} K_1(0,\ 1)\,\varphi(1)\,K_2(1,\ 2)\,ds_1$$

est symétrique, les fonctions fondamentales du noyau $\varphi(1)\,K_2(1, 2)$ se trouvent parmi celles du noyau $\varphi(1)\,K_1(1, 2)$.

b) La fonction $F(2,0) = \int_{(S_1)} \frac{\cos(r_{10}n_1)}{r_{10}^2}\,\frac{1}{r_{12}}\,ds_1$ est symétrique; on le démontre en remarquant que $\frac{1}{r_{12}}$ et $\frac{1}{r_{10}}$ sont harmoniques dans D_i, quand les points (2) et (0) se trouvent dans D_e.

La condition (11) signifie que les fonctions fondamentales du noyau $\frac{\cos(r_{10}n_1)}{r_{10}^2}$ sont des fonctions de Le Roy. Si la fonction V_k de Le Roy et orthogonale à $\rho(1)\,S(0, 1)$, elle satisfait à la condition

$$\frac{dV_{ki}}{dn} = -\frac{dV_{ke}}{dn}, \quad \alpha_k = -1.$$

6. En prenant trois points arbitraires (1), (2) et (3) sur S, on peut écrire autant d'équations de la forme $\rho(1)\cos(r_{21}n_2) - \rho(2)\cos(r_{12}n_1) = 0$. En éliminant de ces équations $\rho(1)$, $\rho(2)$ et $\rho(3)$ on obtient l'équation:

$$\cos(r_{21}n_2)\cos(r_{32}n_3)\cos(r_{13}n_1) - \cos(r_{31}n_3)\cos(r_{12}n_1)\cos(r_{23}n_2) = 0, \tag{13}$$

définissant les surfaces parmi lesquelles il faut chercher celles où les fonctions de Le Roy coïncident avec les fonctions de Poincaré. La condition (13) est suffisante pour les surfaces convexes (sans parties planes). En effet, la condition (13) étant réalisée, et la valeur $f(z)$ étant donnée, trois équations de la forme: $f(1)\cos(r_{21}n_2) - f(2)\cos(r_{12}n_1) = 0$, déterminent sur S une fonction continue, ne changeant pas de signe et ne s'annulant pas. Cette fonction est égale à ρ à un facteur constant près.

On le démontre, en comparant les deux égalités:

$$\frac{1}{2\pi}\int_{(S_1)} \frac{\cos(r_{10}n_1)}{r_{10}^2}\,ds_1 = \frac{1}{2\pi}\int_{(S_1)} f(1)\,S(1, 0)\,ds_1 = 1$$

et

$$f(0) = \frac{1}{2\pi} \int_{(S_1)} f(0)\, S(0, 1)\, f(1)\, ds_1 = \frac{1}{2\pi} \int_{(S_0)} f(1)\, \frac{\cos(r_{01} n_0)}{r_{01}^2}\, ds_1.$$

7. Stekloff a étudié les potentiels de simple couche, vérifiant sur S la condition

$$\frac{dV_{ki}}{dn} = \lambda_k \varphi V_k, \quad \varphi > 0; \quad k = 0, 1, 2, \ldots$$

Il est aisé de voir que, si parmi les systèmes de Stekloff, Poincaré et Le Roy il y en a deux quelconques qui coïncident, le troisième leur est identique. On en conclut, qu'en général les fonctions de Stekloff sont distinctes des fonctions Poincaré et de celles de Le Roy.

Institut de mathématiques et de mécanique de l'Université de Léningrad.

Manuscrit reçu le 13. V. 1937.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

MATHÉMATIQUES

SUR LA SOLUTION EFFECTIVE DE QUELQUES PROBLÈMES LIMITES POUR LES FONCTIONS HARMONIQUES

Par M. KELDYŠ (KELDYSCH) et L. SEDOV

(Présenté par S. Tchaplyguin, de l'Académie, le 16. V. 1937)

Dans cet article nous allons établir une série de formules, donnant la solution effective de quelques problèmes limites pour l'équation de Laplace, ayant des applications dans l'hydrodynamique et la théorie d'élasticité.

§ 1. Il s'agit de déterminer une fonction $\omega(z) = u - iv$ de la variable complexe $z = x + \iota y$, définie et régulière dans le domaine $Imz < 0$, bornée à l'infini et vérifiant les conditions limites suivantes: sur les intervalles $a_k b_k$ de l'axe réel (où $-\infty < a_1 < b_1 < \ldots < b_n = b_0 < +\infty$) les valeurs de la partie imaginaire de v sont données: $v = v_0(x)$, et sur les intervalles $b_{k-1}a_k$ on connait les valeurs de la partie réelle $u = u_0(x)$.

En général, le problème posé n'a pas de solution régulière et bornée dans tout le domaine $Imz < 0$. Les conditions pour l'existence d'une solution bornée, ainsi que la formule donnant cette solution ont été établies par Volterra (1) et Signorini (2).

Dans la plupart des applications ces conditions ne sont pas vérifiées et on doit déterminer une solution, telle que l'intégrale $\int_{-i}^{z} \omega(z)\,dz$ reste bornée au voisinage de chaque point de l'axe réel.

La solution de ce problème est toujours possiblie et dépend de $n+1$ constantes arbitraires. Elle s'exprime par la formule suivante:

$$\omega(z) = \frac{1}{2\pi i}\frac{C_0 + C_1 z + \ldots + C_n z^n}{\sqrt{\prod_{s=1}^{n}(z-a_s)(z-b_s)}} - \frac{1}{\pi i}\sqrt{\prod_{s}\frac{z-a_s}{z-b_s}}\int_{b_0}^{a_1}\sqrt{\prod_{s}\frac{x-b_s}{x-a_s}}\,\frac{u_0(x)\,dx}{x-z} +$$

$$+\frac{1}{\pi i\sqrt{\prod_{s}(z-a_s)(z-b_s)}}\left[\sum_{k=1}^{k=n}\int_{a_k}^{b_k}\frac{v_0\sqrt{(b_k-x)(x-a_k)\prod_{s\neq k}(x-a_s)(x-b_s)}}{x-z}\,dx - \right.$$

$$\left. -\sum_{k=2}^{n}\int_{b_{k-1}}^{a_k}\frac{u_0\sqrt{\prod_{s}(x-a_s)(x-b_s)}\,dx}{x-z}\right], \qquad (1)$$

$C_0, C_1, \ldots, C_n$ étant des constantes réelles arbitraires. Le premier terme de cette formule donne la solution générale du problème homogène correspondant, ce qui correspond au cas $u_0(x) = \varphi_0(x) = 0$.

La solution du problème analogue pour un domaine arbitraire peut être obtenue de la formule (1) si l'on fait une représentation conforme du domaine donné sur le demi-plan inférieur.

Dans l'hydrodynamique il faut souvent trouver une solution du problème considéré, fini en tous les point a_k et s'annulant à l'infini. Dans ce cas on peut déterminer toutes les constantes C_k, ce qui réduit la formule (1) à la forme suivante:

$$\omega_1(z) = \frac{1}{\pi i}\sqrt{\prod_s \frac{z-a_s}{z-b_s}}\left[\sum_{k=1}^{n}\int_{a_k}^{b_k}\sqrt{\frac{b_k-x}{x-a_k}\prod_{s\neq k}\frac{x-b_s}{x-a_s}}\,\frac{\varphi_0(x)\,dx}{x-z} - \right.$$

$$\left. - \sum_{k=1}^{n}\int_{b_{k-1}}^{a_k}\sqrt{\prod_s \frac{x-b_s}{x-a_s}}\,\frac{u_0(x)\,dx}{x-z}\right]. \qquad (2)$$

Dans quelques applications il s'agit de déterminer une fonction $\omega(z)$, vérifiant les mêmes conditions limites et ayant en un nombre fini de points du demi-plan inférieur des pôles, dont les parties principales sont données. Il est facile de voir qu'il suffit de déterminer une telle fonction dans le cas où $u_0(x) = \varphi_0(x) = 0$. La solution $\tilde{\omega}(z)$ de ce problème, ayant un pôle au point $z = z_0$ s'exprime par la formule:

$$\tilde{\omega}(z) = \sqrt{\prod_s \frac{z-a_s}{z-b_s}} \cdot \sum_{j=1}^{m}\left[\frac{k_j}{(z-z_0)^j} - \frac{\bar{k}_j}{(z-\bar{z}_0)^j}\right], \qquad (3)$$

k_j étant les coefficients de la partie principale au point $z = z_0$ de la fonction

$$\tilde{f}(z) = \tilde{\omega}(z)\sqrt{\prod_s \frac{z-b_s}{z-a_s}}. \qquad (4)$$

Indiquons encore que dans le cas où les fonctions $u_0(x)$ et $\varphi_0(x)$ s'expriment par des polynômes dans tous les intervalles $a_k b_k$, $b_{k-1}a_k$, la solution du problème posé au commencement de ce paragraphe peut être représentée par une intégrale hyperelliptique indéfinie (3).

§ 2. Soit D_n un domaine multiplement connexe, limité par n segments de l'axe réel: $a_k \leqslant x \leqslant b_k$ $(k = 1, 2, \ldots, n)$. Considérons le problème de déterminer une fonction $\omega(z)$ de la variable complexe z, satisfaisant aux conditions suivantes: 1) $\omega(z) = u - iv$ est uniforme et régulière dans le domaine D_n; 2) $\omega(z)$ s'annule à l'infini; 3) l'intégrale $\int \omega(z)\,dz$ est bornée aux voisinages des segment $a_k b_k$; 4) $\varphi(x, y)$ tend vers des valeurs limites $\varphi_1(x)$ et $\varphi_2(x)$ respectivement sur le bord supérieur et inférieur des segments mentionnés.

Remarquons que, si les donnés limites sont arbitraires, le problème posé n'a pas de solution bornée aux voisinages des points a_k, b_k.

En posant

$$\left.\begin{aligned}\omega_1(z) &= \frac{\omega(z)+\overline{\omega(\bar{z})}}{2},\\ \omega_2(z) &= \frac{\omega(z)-\overline{\omega(\bar{z})}}{2},\end{aligned}\right\} \tag{5}$$

on s'assure aisément que la détermination de la fonction $\omega_1(z)$ dans le demi-plan inférieur se réduit au problème considéré au commencement du § 1, avec les donnés

$$u_0 = 0, \qquad v_0 = \frac{v_1+v_2}{2};$$

d'autre part on sait les valeurs limites de la partie imaginaire de la fonction $\omega_2(z)$ sur tout l'axe réel. En effet, elles sont données par les formules:

sur les segments $a_k b_k$: $\quad -Im\omega_2(z) = \frac{v_1-v_2}{2}$,

sur les segments $b_{k-1}a_k$: $\quad Im\omega_2(z) = 0$.

Ceci conduit à la formule suivante:

$$\omega(z) = \omega_1(z) + \omega_2(z) = \frac{C_0 + C_1 z + \ldots + C_{n-1}z^{n-1}}{2\pi i}\sqrt{\prod_s (z-a_s)(z-b_s)} +$$

$$+ \frac{1}{2\pi i}\sqrt{\prod_s \frac{z-a_s}{z-b_s}}\sum_{k=1}^{n}\int_{a_k}^{b_k}\sqrt{\frac{b_k-x}{x-a_k}\prod_{s\neq k}\frac{x-b_s}{x-a_s}}\,\frac{v_1+v_2}{x-z}\,dx + \frac{1}{2\pi}\sum_{k=1}^{n}\int_{a_k}^{b_k}\frac{v_1-v_2}{x-z}\,dx. \tag{6}$$

§ 3. Considérons le problème de Dirichlet pour le domaine D_n, défini dans le § 2. Il s'agit de déterminer une fonction harmonique Ψ définie et régulière dans D_n et dont les valeurs limites sur les frontières de D_n sont données *.

Soit φ la fonction harmonique conjuguée avec ψ; nous posons

$$w(z) = \varphi + i\psi \quad \text{et} \quad \omega(z) = \frac{dw}{dz} = u - iv.$$

Les valeurs limites de la fonction v sont connues: $v = -\frac{\partial\psi}{\partial x}$; par conséquent, la fonction $\omega(z)$ peut être calculée par la formule (6). Le développement de la fonction $\omega(z)$ à l'infini doit commencer par un terme du deuxième ordre, d'où

$$C_{n-1} = \sum_{k=1}^{n}\int_{a_k}^{b_k}\sqrt{\frac{b_k-x}{x-a_k}\prod_{s\neq k}\frac{x-b_s}{x-a_s}}\,(v_1+v_2)\,dx. \tag{7}$$

Toutes les autres $n-1$ constantes C_j et la constante additive d'intégration doivent vérifier le système d'équations

$$Im\int_0^{a_k}\omega(z)\,dz = \psi(a_k) - C \qquad (k = 1, 2, \ldots, n). \tag{8}$$

* Nous supposons que ces valeurs limites sont continues et admettent une dérivée continue.

La fonction $w(z)$ est définie par la formule

$$w(z) = \int_0^z \omega(z)\, dz + C. \tag{9}$$

Il ne reste qu'à établir que la fonction ϕ est uniforme et que le système (8) a une solution unique.

Le premier fait résulte de ce que les valeurs limites de ϕ sont uniformes et de la relation (7).

Démontrons la compatibilité et l'indépendance des équations du système (8). Le système d' équations homogènes correspondant est équivalent au système d'équations:

$$\int_{b_k}^{a_{k+1}} \frac{C_0 + C_1 z + \ldots + C_{n-2} z^{n-2}}{\sqrt{\prod_s (z - a_s)(z - b_s)}}\, dz = 0 \quad (k = 1, 2, \ldots, n-1). \tag{10}$$

Il résulte de ces relations que le polynôme $C_0 + C_1 z + \ldots + C_{n-2} z^{n-2}$ a une racine sur chaque segment $b_k a_{k+1}$ et par conséquent $C_0 = C_1 = \ldots = C_{n-2} = 0$ parce qu'un polynôme de dégrè $n-2$ ne peut avoir $n-1$ racines. Il en suit que le déterminant de notre système est nul.

§ 4. Le problème de Neumann pour une fonction harmonique φ dans le domaine D_n se réduit encore à la détermination de la fonction

$$\omega(z) = \frac{dw}{dz} = u - iv, (w = \varphi + i\phi)$$

dont la partie imaginaire $v = \frac{\partial \varphi}{\partial y}$ est donnée sur les segments $a_k b_k$.

On peut déterminer les constantes C_j de telle manière que la fonction φ aît des périodes données, correspondant aux segments $a_k b_k$. On démontre par des raisonnements analogues à ceux du § 3 que le système d' équations correspondantes est résoluble.

Institut Central Aéro-Hydrodynamique.
Moscou.

Manuscrit reçu
le 16. V. 1937.

LITTÉRATURE CITÉE

[1] Volterra, Annali di Matematica, 2-e série, **11**, 1—55, [2] Signorini, Annali di Matematica, 3-e série, **25**, 253—273. [3] M. Lavrentieff et M. Keldysch, Travaux de l'Inst. Central Aéro-Hydrodynamique, N. 152 (1935).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

MATHEMATIK

ÜBER EINIGE ABSCHÄTZUNGEN BEI PSEUDOKONFORMEN ABBILDUNGEN

Von STEFAN BERGMANN

(Vorgelegt von I. M. Vinogradow, Mitglied der Akademie, d. 8. VI. 1937)

Das Schwarzsche Lemma und die Heranziehung der Modulfunktion erlauben in der T. d. a. F. 1 k. V. (Theorie der analytischen Functionen einer komplexen Veränderlichen) Abschätzungen bei konformen Abbildungen, die drei Werte auslassen, zu gewinnen. Die Aufstellung der Riemannschen invarianten Metrik, ihr Zusammenhang mit gewissen Minimalaufgaben und die Einführung der s. g. Vergleichskörper (a, § 1, § 3; b, § 1*), erlaubt die erwähnten Methoden auf den Fall d. a. F. v. 2 k. V. zu übertragen und analoge Abschätzungen bei p. A. (pseudokonformen Abbildungen):

$$\mathbf{W} \qquad w_k = w_k(z_1, z_2), \qquad k = 1, 2, \tag{1}$$

zu gewinnen, die die sechs Ebenen $w_k = 0, = 1, = \infty$, $k = 1, 2$, auslassen. [Eine umkehrbar eindeutige Abbildung eines Bereiches durch ein paar v. a. F. v. 2 k. V. auf einen anderen (ev. in einem mehrblättrigen Raum gelegenen) Bereich bezeichnet man als eine p. A.].

Beim Operieren mit Mengen** sollen die weiter angegebenen Bezeichnungen folgende Bedeutung haben: $\mathfrak{P} \subset \mathfrak{B}$, wenn $\mathfrak{P}$ ein Teilbereich von $\mathfrak{B}$ ist; $\mathfrak{P} < \mathfrak{B}$, wenn jeder Punkt von $\mathfrak{P}$ in $\mathfrak{B}$ liegt***; $\mathfrak{P} \sim \mathfrak{B}$, wenn eine p. A. existiert, die $\mathfrak{P}$ in $\mathfrak{B}$ überführt. Mit $\mathfrak{B}$ werden wir im folgenden stets einen Bereich bezeichnen, für den die Kernfunktion existiert (a, § 1).

§ 1. Wir wollen zunächst einige einfache Verallgemeinerungen des Schwarzschen Lemma auf den Fall v. 2. k. V. und einige sich daraus ergebende Folgerungen besprechen. Im Falle d. a. F. 1. k. V. besagt das Schwarzsche Lemma:

---

* a, § 1.=vgl. dazu Arbeit a, § 1.

** Die Mengen werden stets mit deutschen Buchstaben bezeichnet; für $n < 4$ gibt der obere Index die Dimension der betreffenden Mannigfaltigkeit an.

*** Ist $\mathfrak{P} < \mathfrak{B}$, so können einem Punkte von $\mathfrak{B}$ mehrere Punkte von $\mathfrak{P}$ mit denselben Koordinaten z_1, z_2 entsprechen. Ist $\mathfrak{P} \subset \mathfrak{B}$, so folgt aus der Schlichtheit von $\mathfrak{B}$ die Schlichtheit von $\mathfrak{P}$, ist $\mathfrak{P} < \mathfrak{B}$, so braucht dies nicht mehr der Fall zu sein.

Bildet $w = w(z)$ *den einfachzusammenhängenden Bereich* $\mathfrak{B}^2$ *auf* $\mathfrak{G}^2$ *ab,* $\mathfrak{G}^2 < \mathfrak{B}^2$, *so ist*

$$ds_{\mathfrak{D}^2}(z) = ds_{\mathfrak{E}^2}(w) \geqslant ds_{\mathfrak{B}^2}(w). \tag{2}$$

Führt man die Kernfunktion $K_{\mathfrak{B}^2}(w, \bar{w})$ des Gebietes $\mathfrak{B}^2$ ein (a, die Vorrede), und bildet mit ihrer Hilfe die gegenüber konformen Abbildungen invariante Metrik $ds^2_{\mathfrak{B}^2} = K_{\mathfrak{B}^2}(w, \bar{w})\,|dw|^2$, so stimmt sie im Falle der einfachzusammenhängenden Bereiche mit der üblichen hyperbolischen Metrik des Einheitskreises überein.

Ist $\mathfrak{G}^2 \subset \mathfrak{B}^2$, so folgt (2) aus der Ungleichung a, (1.12) für die Kernfunktionen zweier Gebiete $\mathfrak{G}^2$ und $\mathfrak{B}^2$, $\mathfrak{G}^2 \subset \mathfrak{B}^2$.

Bezeichnet man nun im Falle des vierdimensionalen Raumes die Grösse

$$d\Omega^{(4)}_{\mathfrak{B}}(w_1, w_2) = K_{\mathfrak{B}}(w_1, w_2; \bar{w}_1, \bar{w}_2)\, d\omega^{(4)}(w_1, w_2) \tag{3}$$

als das nichteuklidische Volumen, wo $K_{\mathfrak{B}}$ die Kernfunktion des Gebietes $\mathfrak{B}$ und $d\omega^{(4)}(w_1, w_2)$ das euklidische Volumenelement bedeutet, und

$$d\Omega^{(2)}_{\mathfrak{R}}(w_1, w_2) = \sqrt{K_{\mathfrak{B}}(w_1, w_2; \bar{w}_1, \bar{w}_2)}\, \left|\frac{D(w_1, w_2)}{D(u_1, u_2)}\right| du_1 du_2 \tag{4}$$

als den nichteuklidischen Flächeninhalt eines Elementes der Fläche $w_k = w_k(u_1, u_2)$, u_k reell, $k = 1, 2^*$, so ist—a, (1.15) entsprechend—$d\Omega^{(n)}_{\mathfrak{B}}(w_1, w_2)$, $n = 2, 4$, gegenüber den p. A. invariant. Aus a, (1.12) folgt in diesem Falle:

Ist $\{w_1, w_2\} \subset \mathfrak{P} \subset \mathfrak{B}$, so ist

$$d\Omega^{(n)}_{\mathfrak{P}}(w_1, w_2) \geqslant d\Omega^{(n)}_{\mathfrak{B}}(w_1, w_2), \quad n = 2, 4. \tag{5}$$

Es sei $W = \lambda(w)$ die Modulfunktion, die die in den Punkten 0, 1, ∞ logarithmisch verzweigte Fläche auf den Einheitskreis abbildet und $w = \zeta(W)$, $\zeta'(0) > 0$ ihre Umkehrfunktion. Es sei ferner $\mathfrak{G}$ dasjenige Gebiet, das vermittelst der p. A. $w_k = \zeta(W_k)$ aus $|W_k| < 1$, $k = 1, 2$, entsteht.

Die p. A. (1) *möge* $\mathfrak{B}$ *auf einen Bereich* $\mathfrak{H}$, $\mathfrak{H} \subset \mathfrak{G}$, *abbilden. Es gilt dann:*

$$d\omega^{(4)}(w_1, w_2) \leqslant G_4(w_1, w_2)\, d\Omega^{(4)}_{\mathfrak{B}}(z_1, z_2), \tag{6}$$

$$\left|\frac{D(w_1, w_2)}{D(u_1, u_2)}\right| du_1 du_2 \leqslant G_2(w_1, w_2)\, d\Omega^{(2)}_{\mathfrak{B}}(z_1, z_2), \tag{7}$$

---

* (3) ist das Volumenelement der Metrik (11), wenn man

$$I = \sqrt{\frac{K}{T_{1\bar{1}}\, I\, T_{2\bar{2}} - |T_{1\bar{2}}|^2}}$$

setzt.

Wie mich B. Fuchs darauf aufmerksam gemacht hat, gilt in diesem Falle:

$$d\Omega^{(4)}(w_1, w_2) = \frac{1}{2}\, ds_1 ds_2 \sin F,\ ds_k^2 \equiv \sum_{m,\, n=1}^{2} T_{m\bar{n}} \frac{\partial w_m}{du_k} \frac{d\bar{w}_n}{du_k}\, du_k^2,$$

wo F den Winkel zwischen den analytischen Ebenen bedeutet, in denen die Vektoren $\left\{\frac{\partial w_1}{\partial u_k}, \frac{\partial w_2}{\partial u_k}\right\}$, $k = 1, 2$, liegen [siehe (2), § 1].

worin

$$G_n(w_1, w_2) = \prod_{k=1}^{2} \left\{ \frac{\sqrt{\pi}\,[1 - |\lambda(w_k)|^2]}{|\lambda'(w_k)|} \right\}^{\frac{n}{2}}$$

bedeutet.

(6) und (7) folgen aus (3), (4), (5), wenn man berücksichtigt, dass laut a, (4.11) $K_{\mathfrak{C}}(w_1, w_2; \bar{w}_1, \bar{w}_2) = \prod_{k=1}^{2} \left\{ \frac{|\lambda'(w_k)|^2}{\pi\,[1 - |\lambda(w_k)|^2]^2} \right\}$ gilt.

Korollar 1. Aus (6) folgt

$$\left| \frac{D(w_1, w_2)}{D(z_1, z_2)} \right|^2 = \frac{d\omega^{(4)}(w_1, w_2)}{d\omega^{(4)}(z_1, z_2)} \leqq G_4(w_1, w_2)\, K_{\mathfrak{B}}(z_1, z_2; \bar{z}_1, \bar{z}_2). \tag{8}$$

Korollar II. Ordnet man dem Flächenelement die Grösse

$$\Phi = \left| \frac{D(w_1, w_2)}{D(u_1, u_2)} \right| : \begin{vmatrix} \Sigma \varphi_{k1}^2 & \Sigma \varphi_{k1} \varphi_{k2} \\ \Sigma \varphi_{k1} \varphi_{k2} & \Sigma \varphi_{k2}^2 \end{vmatrix}, \tag{9}$$

$$w_1 = \varphi_1 + i\varphi_2,\ w_2 = \varphi_3 + i\varphi_4,\ \varphi_{kn} \equiv \frac{\partial \varphi_k}{\partial u_n},\ \sum \equiv \sum_{k=1}^{4},$$

(c, § 2) zu, so ist:

$$d\omega^{(2)}(w_1, w_2) \leqq G_2(w_1, w_2)\, \Phi^{-1} d\Omega_{\mathfrak{B}}^{(2)}(z_1, z_2). \tag{10}$$

§ 2. In diesem Paragraphen geben wir einige Anschätzungen für

$$ds_{\mathfrak{H}}(w_1, w_2) = I \sum_{m,n=1}^{2} T_{mn} dw_m d\bar{w}_n,\ T_{m\bar{n}} = \frac{\partial^2 \lg K_{\mathfrak{B}}(w_1, w_2, \bar{w}_1, \bar{w}_2)}{\partial w_m \partial \bar{w}_n},\ I = 1 \tag{11}$$

bei p. A. von $\mathfrak{B}$ auf $\mathfrak{H}$, $\mathfrak{H} < \mathfrak{G}$.

Ist $\mathbf{W}(\mathfrak{E}) = \mathfrak{H}$, $\mathfrak{H} < \mathfrak{E}$, *wo* $\mathfrak{E}$ *den Bizylinder* $|z_k| < 1$, $k = 1, 2$, *bedeutet, so ist*

$$ds_{\mathfrak{E}}(w_1, w_2) \leqq ds_{\mathfrak{H}}(w_1, w_2) = ds_{\mathfrak{E}}(z_1, z_2). \tag{12}$$

Um (12) zu beweisen, zeigt man zunächst, dass bei einer mittelpunkttreuen p. A. $w_k = f_k(z_1, z_2) = \alpha_{k1} z_1 + \alpha_{k2} z_2 + \ldots$, $k = 1, 2$, von $\mathfrak{E}$ auf $\mathfrak{H}$ jede Hyperkugel $\mathfrak{K}[|z_1|^2 + |z_2|^2 < r^2, r < 1]$ in einen Bereich $\mathfrak{N}$, $\mathfrak{N} < \mathfrak{K}$ übergeht*. Betrachten wir einen Punkt $\{z_1, z_2\}$, er liegt auf der analytischen Ebene $z_2 = z_1 m$, $m = \frac{z_2}{z_1}$. Für die Funktion $f_1(z_1, mz_1) = z_1(\alpha_{11} + m\alpha_{12}) + \ldots$ gilt $|f_1(z_1, mz_1)| \leqq 1$ für $|z_1| \leqq 1$, woraus $|f_1(z_1, mz_1)| \leqq |z_1|$ folgt. Ebenso zeigt man dass $|f_2(Z_1, Z_2)| \leqq |Z_2|$ gilt. Da die Indikatrix $\sum_{m,n=1}^{2} T_{m\bar{n}}(z_1 z_2)$, $Z_m \bar{Z}_n < 1$ der Metrik von $\mathfrak{E}$, im Punkte $\{z_1, z_2\} = \{0, 0\}$ eine Hyperkugel ist, so folgt aus

$$|f_1(Z_1, Z_2)|^2 + |f_2(Z_1, Z_2)|^2 \leqq |Z_1|^2 + |Z_2|^2:$$

$$ds_{\mathfrak{H}}(0, 0) \geqq ds_{\mathfrak{E}}(0, 0). \tag{13}$$

---

* Der Beweis dieser letzten Behauptung wird ganz analog einem Beweis von K. Reinhardt (3) durchgeführt.

Da man $\mathfrak{E}$ so in sich abbilden kann, dass $\{0, 0\}$ in einen vorgegebenen Punkt von $\mathfrak{E}$ übergeht, folgt (12), aus (13) und dem

Hilfssatz. *Es möge* $\mathfrak{L}$ *durch eine p. A.* $\mathbf{H}$, $\mathbf{H}(0, 0) = \{0, 0\}$ aus $\mathfrak{E}$ *entstehen und möge die p. A.* $\mathbf{S}$, $\mathbf{S}(0, 0) = \{0, 0\}$, $\mathfrak{L}$ *in* $\mathfrak{T}$, $\mathfrak{T} < \mathfrak{L}$ *überführen. Dann ist:* $ds_{\mathfrak{T}}(0, 0) \geqslant ds_{\mathfrak{L}}(0, 0)$ (d, Hilfssatz 3, S. 94).

Bemerkung. Aus dem Hilfssatz folgt, dass (12) auch dann gilt, wenn wir unter $\mathfrak{E}$ irgendeinen Bereich verstehen, der durch eine p. A. aus dem Bizylinder entsteht.

Satz. *Es sei* $\mathfrak{V} \subset \mathfrak{B}$, $\mathfrak{V} \sim \mathfrak{E}$, *und möge* $ds^2_{\mathfrak{V}}(z_1, z_2) \leqslant C(z_1, z_2)\, ds^2_{\mathfrak{B}}(z_1, z_2)$ *gelten , wo C eine geeignete Grösse ist. Bei einer p. A* (I) *von* $\mathfrak{V}$ *auf* $\mathfrak{H}$, $\mathfrak{H} < \mathfrak{G}$ *ist*

$$\sum_{k=1}^{2} \frac{2\,|\lambda'(w_k)|^2\,|dw_k|^2}{\pi^2[1-|\lambda(w_k)|^2]^2} \leqslant C(z_1, z_2)\, ds^2_{\mathfrak{B}}(z_1\ z_2). \tag{14}$$

Beweis. Es sei $\mathbf{R}$, $z_k = z_k(Z_1, Z_2)$, $k = 1, 2$, diejenige p. A., die $\mathfrak{G}$ auf $\mathfrak{V}$ abbildet. $\mathbf{WR}$ bildet dann $\mathfrak{G}$ auf $\mathfrak{T} \subset \mathfrak{H} < \mathfrak{G}$ ab. Da $\mathfrak{G} \sim \mathfrak{E}$ ist, so gilt

$$ds^2_{\mathfrak{E}}(w_1, w_2) \leqslant ds^2_{\mathfrak{E}}(Z_1, Z_2) = ds^2_{\mathfrak{V}}(z_1, z_2) \leqslant C(z_1, z_2)\, ds^2_{\mathfrak{B}}(z_1, z_2).$$

Laut a, (4.11) ist aber

$$ds^2_{\mathfrak{E}}(w_1, w_2) = \sum_{k=1}^{2} \frac{2\,|\lambda'(w_k)|^2\,|dw_k|^2}{\pi^2[1-|\lambda(w_k)|^2]^2}.$$

Indem man (14) integriert, erhält man zu jedem Gebiet $\mathfrak{D} \subset \mathfrak{B}$ und zu den Punkten $\{a_1, a_2\} \subset \mathfrak{D}$ und $\{A_1, A_2\}$ einen Bereich $\mathfrak{M}$ der jedes $\mathbf{W}(\mathfrak{D})$ enthält, wobei $\mathbf{W}$, $\mathbf{W}(a_1, a_2) = \{A_1, A_2\}$ eine beliebige p. A. von $\mathfrak{B}$ bedeutet, die die sechs angegebenen Ebenen auslässt.

Mathematisches Institut der Georgischen
Filiale der Akademie der Wissenschaften
der UdSSR.

Eingegangen
d. 8. VI 1937

ZITIERTE LITERATUR

[1] a. Crelles Journal f. reine u. ang. Math, **169**, 1—42 (1933); b. Ibidem **172**, 89—128 (1934); c. Rendiconti d. r. Accademia naz. d. Lincei **19** (6 ser). 477—478 (1934); d. Recueil Mathém., **1 (43)**, 79—96, (1936). [2] Recueil Mathém., **1 (43)** 561—574 (1936). [3] Math. Annalen, **83**, 216—255 (1921).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

MECHANICS

THE BOUNDARY DYNAMIC PROBLEM OF THE THEORY OF ELASTICITY FOR A CIRCLE WITH GIVEN DISPLACEMENTS

By J. A. MINDLIN

(Communicated by I. M. Vinogradow, Member of the Academy, 7. V. 1937)

The present note renders a solution of the problem of oscillation of a circle with a radius R, at initial zero conditions and at given projections on the circumference u_r, u_θ of the displacement vector.

Projections of the vectors of displacement in the polar coordinates are expressed by the formulae:

$$u_r = \frac{\partial \varphi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}; \quad u_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi}{\partial r}, \tag{1}$$

where functions φ and ψ satisfy the equations:

$$\frac{1}{a^2}\frac{\partial^2 \varphi}{\partial t^2} = \Delta\varphi; \quad \frac{1}{b^2}\frac{\partial^2 \psi}{\partial t^2} = \Delta\psi; \tag{2}$$

$$a = \sqrt{\frac{\lambda + 2\mu}{\rho}}; \quad b = \sqrt{\frac{\mu}{\rho}}; \tag{3}$$

by ρ we denote density of the medium, while λ and μ are elastic constants of Lamé.

Assuming that the boundary conditions have the following form:

$$u_r|_{r=R} = \alpha_n(t)\, e^{in\theta}, \quad u_\theta|_{r=R} = \beta_n(t)\, e^{in\theta}, \tag{4}$$

where n is an integer number or zero.

In order to find the sought functions φ_n and ψ_n we apply the principle of superposition(1).

We have:

$$\varphi_n = \int_0^{2\pi} f_n(x\cos\lambda + y\sin\lambda + at)\, e^{in\lambda}\, d\lambda,$$

$$\psi_n = \int_0^{2\pi} F_n(x\cos\lambda + y\sin\lambda + bt)\, e^{in\lambda}\, d\lambda.$$

The expressions of φ_n and ψ_n in the polar coordinates are

$$\varphi_n = \int_0^{\pi} A_n(r\cos\xi + at)\cos n\,\xi d\xi e^{in\theta}, \quad \psi_n = \int_0^{\pi} B_n(r\cos\xi + bt)\cos n\,\xi d\xi e^{in\theta}.$$

Satisfying the boundary conditions (4), after a silmple transformation, we obtain a system of integral equations for the determination of the complex functions $A_n(\tau)$, $B_n(\tau)$:

$$\left.\begin{aligned} &\int_0^{\pi} A_n'(R\cos\xi + at)\cos(n+1)\,\xi d\xi - \\ &-i\int_0^{\pi} B_n'(R\cos\xi + bt)\cos(n+1)\,\xi d\xi = \alpha_n(t) + i\,\beta_n(t), \\ &\int_0^{\pi} A_n'(R\cos\xi + at)\cos(n-1)\,\xi\, d\xi + \\ &+i\int_0^{\pi} B_n'(R\cos\xi + bt)\cos(n-1)\,\xi d\xi = \alpha_n(t) - i\,\beta_n(t). \end{aligned}\right\} \tag{5}$$

In view of the non-contradiction of the initial and boundary conditions, we have:

$$\alpha_n(0) = 0, \quad \alpha_n'(0) = 0; \quad \beta_n(0) = 0, \quad \beta_n'(0) = 0$$

from which it follows that the integral equations (5) at $t = 0$ will be satisfied if we put the functions

$$A_n'(\tau) = 0, \quad B_n'(\tau) = 0, \quad -R \leqslant \tau \leqslant R. \tag{6}$$

Substitute in both sides of the equations (5) the variable t with a new variable x, defined by the equation $x = R + at$, whereupon introduce a new variable of integration $y = R\cos\xi + x - R$ in the first terms of the left sides of the equations (5) while in the second terms put $R\cos\xi + \frac{b}{a}x = \frac{b}{a}y + R$; then for t varying in the interval $0 \leqslant t \leqslant \frac{R}{a}$ (which corresponds to the variation of the x in the interval $R \leqslant x \leqslant 2R$) according to the conditions (6) we shall have:

$$\left.\begin{aligned} &\int_R^x \frac{A_n'(y)\,T_{n-1}\left(\frac{y+R-x}{R}\right)dy}{\sqrt{R^2-(y+R-x)^2}} + \\ &+ i\frac{b}{a}\int_R^x \frac{B_n'\left[\frac{b}{a}y + R\left(1-\frac{b}{a}\right)\right]T_{n-1}\left[\frac{\frac{b}{a}(y-x)+R}{R}\right]dy}{\sqrt{R^2-\left[\frac{b}{a}(y-x)+R\right]^2}} = \\ &= \alpha_n\left(\frac{x-R}{a}\right) - i\,\beta_n\left(\frac{x-R}{a}\right); \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}
&\int_R^x \frac{A_n'(y)\,T_{n+1}\left(\dfrac{y+R-x}{R}\right)dy}{\sqrt{R^2-(y+R-x)^2}} - \\
&-i\frac{b}{a}\int_R^x \frac{B_n'\left[\dfrac{b}{a}y+R\left(1-\dfrac{b}{a}\right)\right]T_{n+1}\left[\dfrac{\dfrac{b}{a}(y-x)+R}{R}\right]dy}{\sqrt{R^2-\left[\dfrac{b}{a}(y-x)+R\right]^2}} = \\
&\qquad = \alpha_n\left(\frac{x-R}{a}\right)+i\,\beta_n\left(\frac{x-R}{a}\right).
\end{aligned}\right\} \qquad (7)$$

The function $T_n(x) = \cos(n \arccos x)$ is the polynomial of Tchebyshev.

Denoting $B_n'\left[\frac{b}{a}y + R\left(1-\frac{b}{a}\right)\right] = \overline{B}_n(y)$ we have: the equations (7) represent a system of the Volterra integral equations of the 1st type with a nucleus depending on the difference of arguments. Performing the Laplace transformation on the equations (7) we multiply both parts by e^{-sx} and integrate from $x = R$ to $x = \infty$.

Assuming that

$$\int_R^\infty e^{-sy} A_n'(y)\,dy = \overset{*}{\varphi}{}^1(s), \qquad \int_R^\infty e^{-sy}\overline{B}_n(y)\,dy = \overset{*}{\varphi}{}^2(s); \qquad (8)$$

$$\int_R^\infty e^{-sx}\left[\alpha_n\left(\frac{x-R}{a}\right) - i\,\beta_n\left(\frac{x-R}{a}\right)\right]dx = f^1(s);$$

$$\int_R^\infty e^{-sx}\left[\alpha_n\left(\frac{x-R}{a}\right) + i\,\beta_n\left(\frac{x-R}{a}\right)\right]dx = f^2(s); \qquad (9)$$

$$\int_R^\infty e^{-sz} K_l^m(z)\,dz = \overset{*}{K}{}_l^m(s) \qquad (l = 1, 2; \quad m = 1, 2), \qquad (10)$$

where functions $\overset{*}{K}{}_l^m(z)$ are nuclei of the system of integral equations (7), we obtain:

$$\overset{*}{K}{}_l^m(s)\,\overset{*}{\varphi}{}^l(s) = f^m(s), \qquad (11)$$

where the determinant of this system

$$\Delta s = \| \overset{*}{K}{}_l^m(s) \|$$

in the considered interval differs from zero.

Let Φ_p^l represent the elements of the matrix $\Delta^{-1}(s)$, inverse of the matrix $\Delta(s)$; the solution of the system (11) is then given by the equation

$$\overset{*}{\varphi}{}^l(s) = \Phi_p^l f^p(s), \qquad (12)$$

wherefrom, by means of the Riemann-Mellin formula, we obtain the functions $A_n'(y)$ and $\overline{B}_n(y)$ of the equations (8):

$$\left.\begin{aligned} A_n'(y) &= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} e^{ys}\Phi_p^1(s)\,f^p(s)\,ds, \\ \overline{B}_n(y) &= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} e^{ys}\Phi_p^2(s)\,f^p(s)\,ds. \end{aligned}\right\} \tag{13}$$

Exactly in the same manner as were determined functions $A_n'(y)$ and $\overline{B}_n(y)$ for $R \leqslant y \leqslant 2R$, we can also determine them for $2R \leqslant y \leqslant 3R$ for which purpose it is merely necessary to take into account the wave's reflection. Turning to the system of integral equations (5) and examining the changes of the variable t in the interval $\frac{R}{a} \leqslant$ $\leqslant t \leqslant \frac{2R}{a}$, we divide the intervals of integration $(-R+at;\ R+at)$ and $(-R+bt;\ R+bt$ (respectively into $(-R+at;\ 2R)$ and $(2R;\ R+at)$;

$$\left[-R+bt;\ R\left(1+\frac{b}{a}\right)\right] \text{ and } \left[R\left(1+\frac{b}{a}\right);\ R+bt\right].$$

Hence to the given boundary conditions will be added, to the left side of the integral equations, the known terms of wave's reflection. The following instants of time are considered analogically. Thus the functions $A_n'(y)$ and $B_n'(y)$ are determined in the entire interval of variation of y. By integrating them we shall find $A_n(y)$ and $B_n(y)$:

$$A_n(y) = \int_{-R}^{y} A_n'(y)\,dy, \quad B_n(y) = \int_{-R}^{y} B_n'(y)\,dy.$$

Without loosing any generality we can set the arbitrary constant of integration equal to zero.

In conclusion, the author feels bound to express his gratitude to S. L. Sobolev, at whose proposal and under whose supervision the present work was accomplished.

Commission of Technical Mathematics.
Group of Technical Mechanics.
Academy of Sciences of the USSR.
Moscow.

Received
7. V. 1937.

REFERENCES

[1] Whittaker, Mathematische Annalen (1903).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

THEORY OF ELASTICITY

ON THE DESIGN OF A SPHERICAL SHELL

By V. SOKOLOVSKIJ

(Communicated by A. Krylov, Member of the Academy, 26. V. 1937)

In this paper, we give the solution of the problem of a design of a thin elastic spherical shell, assuming the deformation to be small.

We employ the notation usual in the theory of elasticity (1) and, for the sake of obtaining the equations of a spherical shell in a form convenient for us, we deduce them, using Love's hypotheses, from the equations of equilibrium in spherical coordinates*:

$$\left.\begin{aligned}
&\frac{\partial\widehat{\theta\theta}}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\widehat{\theta\varphi}}{\partial\varphi}+(\widehat{\theta\theta}-\widehat{\varphi\varphi})\operatorname{ctg}\theta+\widehat{\theta r}+\frac{1}{r}\frac{\partial}{\partial r}(r^2\widehat{\theta r})+r\rho F_\theta=0,\\
&\frac{\partial\widehat{\theta\varphi}}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\widehat{\varphi\varphi}}{\partial\varphi}+2\widehat{\theta\varphi}\operatorname{ctg}\theta+\widehat{\varphi r}+\frac{1}{r}\frac{\partial}{\partial r}(r^2\widehat{\varphi r})+r\rho F_\varphi=0,\\
&\frac{\partial\widehat{\theta r}}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\widehat{\varphi r}}{\partial\varphi}+\widehat{\theta r}\operatorname{ctg}\theta-(\widehat{\theta\theta}+\widehat{\varphi\varphi})+\frac{1}{r}\frac{\partial}{\partial r}(r^2\widehat{rr})+r\rho F_r=0.
\end{aligned}\right\}\qquad(1)$$

For this purpose, let us express the strains $\widehat{\theta\theta}$, $\widehat{\varphi\varphi}$ and $\widehat{\theta\varphi}$ in terms of ω_r, ω_θ, ω_φ, Ω assuming, similarly to Love, that

$$e_{rr}=-\frac{\sigma}{1-\sigma}\Omega,\quad\text{where}\quad\Omega=e_{\theta\theta}+e_{\varphi\varphi}.$$

After appropriate transformations, we obtain:

$$\left.\begin{aligned}
&\frac{E}{1-\sigma^2}\frac{\partial\Omega}{\partial\theta}-\frac{E}{1+\sigma}\cdot\frac{1}{\sin\theta}\cdot\frac{\partial\omega_r}{\partial\varphi}+\frac{E}{1+\sigma}\left[2\omega_\varphi-\frac{\partial u}{\partial r}\right]+\\
&\qquad+\widehat{\theta r}+\frac{1}{r}\frac{\partial}{\partial r}(r^2\widehat{\theta r})+r\rho F_\theta=0,\\
&\frac{E}{1-\sigma^2}\frac{1}{\sin\theta}\frac{\partial\Omega}{\partial\varphi}+\frac{E}{1+\sigma}\frac{\partial\omega_r}{\partial\theta}-\frac{E}{1+\sigma}\left[2\omega_\theta+\frac{\partial v}{\partial r}\right]+\\
&\qquad+\widehat{\varphi r}+\frac{1}{r}\frac{\partial}{\partial r}(r^2\widehat{\varphi r})+r\rho F_\varphi=0,\\
&\frac{\partial\widehat{\theta r}}{\partial\theta}+\frac{1}{\sin\theta}\cdot\frac{\partial\widehat{\varphi r}}{\partial\varphi}+\widehat{\theta r}\operatorname{ctg}\theta-\frac{E}{1-\sigma}\Omega+\frac{1}{r}\frac{\partial}{\partial r}(r^2\widehat{rr})+r\rho F_r=0.
\end{aligned}\right\}\qquad(2)$$

* θ is the width, φ is the length, a is the radius of the middle surface ($r=a-z$).

Love's assumption to the effect that the linear elements perpendicular to the as yet undeformed middle surface remain rectilinear and perpendicular to the middle surface after the deformation, reduces to

$$[e_{\theta r}]_{r=a} = 0, \qquad [e_{\varphi r}]_{r=a} = 0,$$

whence, if the displacements are taken as *

$$u = u^{(0)} + z \cdot u^{(1)}, \quad \varrho = \varrho^{(0)} + z \cdot \varrho^{(1)}, \quad w = w^{(0)} + z \cdot w^{(1)}, \tag{3}$$

it follows that

$$\left.\begin{aligned} u^{(1)} &= \frac{1}{a}\frac{\partial w^{(0)}}{\partial \theta} - \frac{u^{(0)}}{a} = -\Theta, \\ \varrho^{(1)} &= \frac{1}{a \sin\theta}\cdot\frac{\partial w^{(0)}}{\partial \varphi} - \frac{\varrho^{(0)}}{a} = \Phi. \end{aligned}\right\} \tag{4}$$

It should be noted that, owing to (3), we have:

$$\bar{e}_{\theta\theta} = \bar{e}^{(0)}_{\theta\theta} + z \cdot \bar{e}^{(1)}_{\theta\theta}, \quad \bar{e}_{\varphi\varphi} = \bar{e}^{(0)}_{\varphi\varphi} + z \cdot \bar{e}^{(1)}_{\varphi\varphi}, \quad \bar{e}_{\theta\varphi} = \bar{e}^{(0)}_{\theta\varphi} + z \cdot \bar{e}^{(1)}_{\theta\varphi},$$
$$\bar{\omega}_r = \bar{\omega}^{(0)}_r + z \cdot \bar{\omega}^{(1)}_r, \quad \bar{\Omega} = \bar{\Omega}^{(0)} + z\bar{\Omega}^{(1)},$$

where

$$\bar{e}_{\theta\theta} = \frac{r}{a} e_{\theta\theta}, \quad \bar{e}_{\varphi\varphi} = \frac{r}{a} e_{\varphi\varphi}, \quad \bar{e}_{\theta\varphi} = \frac{r}{a} e_{\theta\varphi}, \quad \bar{\omega}_r = \frac{r}{a}\omega_r, \quad \bar{\Omega} = \frac{r}{a}\Omega.$$

Multiplying (2) either by $\frac{r}{a}$ or by $z\frac{r}{a}$, integrating with respect to z from $-h$ to $+h$ and noting that

$$\int_{-h}^{+h} \frac{r}{a}\left(2\omega_\varphi - \frac{\partial u}{\partial r}\right) dz = \int_{-h}^{+h}\left(\frac{u}{a} - \frac{1}{a}\frac{\partial w}{\partial \theta}\right) dz = \left(\frac{u^{(0)}}{a} - \frac{1}{a}\cdot\frac{\partial w^{(0)}}{\partial \theta}\right) 2h = \Theta \cdot 2h,$$

$$\int_{-h}^{+h} z\frac{r}{a}\left(2\omega_\varphi - \frac{\partial u}{\partial r}\right) dz = -\frac{\Theta}{a}\cdot\frac{2h^3}{3}, \quad \int_{-h}^{+h}\frac{r}{a}\left(2\omega_\theta + \frac{\partial \varrho}{\partial r}\right) dz = \Phi \cdot 2h,$$

$$\int_{-h}^{+h} z\frac{r}{a}\left(2\omega_\theta + \frac{\partial \varrho}{\partial r}\right) dz = -\frac{\Phi}{a}\frac{2h^3}{3},$$

we obtain the equation of equilibrium of a spherical shell in the form **:

$$\left.\begin{aligned} B\frac{\partial \bar{\Omega}^{(0)}}{\partial \theta} - B(1-\sigma)\frac{1}{\sin\theta}\frac{\partial \bar{\omega}^{(0)}_r}{\partial \varphi} + B(1-\sigma)\Theta - N_1 + aX &= 0, \\ B\frac{1}{\sin\theta}\frac{\partial \bar{\Omega}^{(0)}}{\partial \varphi} + B(1-\sigma)\frac{\partial \bar{\omega}^{(0)}_r}{\partial \theta} - B(1-\sigma)\Phi - N_2 + aY &= 0, \\ \frac{\partial N_1}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial N_2}{\partial \varphi} + \operatorname{ctg}\theta N_1 + B(1+\sigma)\bar{\Omega}^{(0)} + aZ &= 0, \\ D\frac{\partial \bar{\Omega}^{(1)}}{\partial \theta} - D(1-\sigma)\frac{1}{\sin\theta}\frac{\partial \bar{\omega}^{(1)}_r}{\partial \varphi} - \frac{D(1-\sigma)}{a}\Theta - aN_1 + a\mathfrak{M} &= 0, \\ D\frac{1}{\sin\theta}\frac{\partial \bar{\Omega}^{(1)}}{\partial \varphi} + D(1-\sigma)\frac{\partial \bar{\omega}^{(1)}_r}{\partial \theta} + \frac{D(1-\sigma)}{a}\Phi - aN_2 - a\mathfrak{L} &= 0. \end{aligned}\right\} \tag{5}$$

* $u^{(0)}$, $\varrho^{(0)}$, $w^{(0)}$ is the displacement of the middle surface along the meridian, the parallel and the outer normal respectively.

** $B = \frac{2Eh}{1-\sigma^2}$, $\qquad D = \frac{2Eh^3}{3(1-\sigma)}$.

For convenience let us use the following notation:

$$l(\ldots) = \frac{\partial(\ldots)}{\partial\theta} + \operatorname{ctg}\theta(\ldots), \quad k(\ldots) = \frac{1}{\sin\theta}\frac{\partial(\ldots)}{\partial\varphi}. \tag{6}$$

Further, it can be easily seen that

$$2\overline{\omega}_r^{(0)} = -[l(\Phi) + k(\Theta)], \tag{7}$$

$$2\overline{\omega}_r^{(1)} = \frac{1}{a}[l(\Phi) + k(\Theta)]. \tag{8}$$

If in the expression

$$\overline{\Omega}^{(1)} = -\frac{1}{a}\left[\{l(\Theta) - k(\Phi)\} - 2w^{(1)}\right],$$

following Love, we omit $w^{(1)}$, which is small as compared with the other term within the brackets *, we obtain:

$$\overline{\Omega}^{(1)} = -\frac{1}{a}[l(\Theta) - k(\Phi)]. \tag{9}$$

Equations (5) may be obtained also directly from the equations of equilibrium of shells [1], if we express in the latter the stresses T_1, T_2, S_1 and moments G_1, G_2, H_1 in terms of the displacements of the middle surface $u^{(0)}$, $v^{(0)}$, $w^{(0)}$, according to formulae known in the theory of shells, and use the notations (4), (7), (8) and (9).

If now, on determining $\overline{\Omega}^{(0)}$ from the third equation of system (5)

$$\overline{\Omega}^{(0)} = -\frac{1}{B(1+\sigma)}[l(N_1) + k(N_2) + aZ], \tag{10}$$

we substitute (7), (8), (9) and (10) in (5), we obtain:

$$\left.\begin{aligned} &\frac{\partial}{\partial\theta}[l(N_1) + k(N_2)] - \frac{B(1-\sigma^2)}{2}k(V) - B(1-\sigma^2)\Theta + \\ &+(1+\sigma)N_1 + a\left[\frac{\partial Z}{\partial\theta} - (1+\sigma)X\right] = 0, \\ &k[l(N_1) + k(N_2)] + \frac{B(1-\sigma^2)}{2}\frac{\partial V}{\partial\theta} + B(1-\sigma^2)\Phi + \\ &+(1+\sigma)N_2 + a[k(Z) - (1+\sigma)Y] = 0; \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} &\frac{\partial U}{\partial\theta} + \frac{1-\sigma}{2}k(V) + (1-\sigma)\Theta + \frac{a^2}{D}N_1 - \frac{a^2}{D}\mathfrak{M} = 0, \\ &k(U) - \frac{1-\sigma}{2}\frac{\partial V}{\partial\theta} - (1-\sigma)\Phi + \frac{a^2}{D}N_2 + \frac{a^2}{D}\mathfrak{L} = 0, \end{aligned}\right\} \tag{12}$$

where

$$l(\Theta) - k(\Phi) = U, \qquad l(\Phi) + k(\Theta) = V. \tag{13}$$

From equations (12) we determine:

$$\left.\begin{aligned} N_1 &= -\frac{D}{a^2}\left[\frac{\partial U}{\partial\theta} + \frac{1-\sigma}{2}k(V) + (1-\sigma)\Theta\right] + \mathfrak{M}, \\ N_2 &= -\frac{D}{a^2}\left[k(U) - \frac{1-\sigma}{2}\frac{\partial V}{\partial\theta} - (1-\sigma)\Phi\right] - \mathfrak{L}. \end{aligned}\right\} \tag{14}$$

---

* Since $w^{(1)} = \frac{\sigma}{1-\sigma}\overline{\Omega}^{(0)}$ is of the order of $h\overline{\Omega}^{(1)}$, Love accordingly omits ζ in the expressions e_{xx} and e_{yy} [1].

Inserting (14) in (11), we obtain *:

$$\left.\begin{aligned} &\frac{\partial}{\partial\theta}[L(U)+U]+\frac{\nu^2+1}{2}k(V)+(\nu^2+1)\Theta+\Psi_1=0,\\ &k[L(U)+U]-\frac{\nu^2+1}{2}\frac{\partial V}{\partial\theta}-(\nu^2+1)\Phi+\Psi_2=0, \end{aligned}\right\} \tag{15}$$

where

$$\left.\begin{aligned} L(\ldots)&=\frac{\partial^3(\ldots)}{\partial\theta^3}+\operatorname{ctg}\theta\frac{\partial(\ldots)}{\partial\theta}+(\ldots)+k^2(\ldots),\\ \nu^2&=\frac{3(1-\sigma^2)a^2}{h^2}-\sigma^2. \end{aligned}\right\} \tag{16}$$

Expressing Θ and Φ in terms of U and V from (15)

$$\left.\begin{aligned} \Theta&=-\frac{1}{\nu^2+1}\frac{\partial}{\partial\theta}[L(U)+U]-\frac{1}{2}k(V)-\frac{1}{\nu^2+1}\Psi_1,\\ \Phi&=\frac{1}{\nu^2+1}k[L(U)+U]-\frac{1}{2}\frac{\partial V}{\partial\theta}+\frac{1}{\nu^2+1}\Psi_2, \end{aligned}\right\} \tag{17}$$

and inserting them in (13) and (14), we obtain:

$$LL(U)+\nu^2\cdot U+[l(\Psi_1)+k(\Psi_2)]=0, \tag{18}$$

$$L(V)+V+\frac{2}{\nu^2+1}[k(\Psi_1)-l(\Psi_2)]=0, \tag{19}$$

$$\left.\begin{aligned} N_1&=\frac{D(1-\sigma)}{a^2(\nu^2+1)}\cdot\frac{\partial}{\partial\theta}[L(U)+U]-\frac{D}{a^2}\frac{\partial U}{\partial\theta}+\frac{D(1-\sigma)}{a^2(\nu^2+1)}\Psi_1+\mathfrak{M},\\ N_2&=\frac{D(1-\sigma)}{a^2(\nu^2+1)}\cdot k[L(U)+U]-\frac{D}{a^2}k(U)+\frac{D(1-\sigma)}{a^2(\nu^2+1)}\Psi_2-\mathfrak{L}. \end{aligned}\right\} \tag{14'}$$

When the edge of the shell consists of two parallels ($\theta=\text{const}$), the forces and couples (external and those acting on edges) acting upon the shell must either be independent of φ or periodic functions of φ with period 2π and we may suppose them developed in Fourier's series of sines and cosines of integral multiples of φ.

We shall write:

$$\left.\begin{aligned} &u^{(0)}=u_n^{(0)}\cos(n\varphi+\varepsilon),\ \Theta=\Theta_n\cos(n\varphi+\varepsilon),\ \overline{\omega}_r^{(0)}=\overline{\omega}_{rn}^{(0)}\sin(n\varphi+\varepsilon),\\ &v^{(0)}=v_n^{(0)}\sin(n\varphi+\varepsilon),\ \Phi=\Phi_n\sin(n\varphi+\varepsilon),\ \overline{\Omega}^{(0)}=\overline{\Omega}_n^{(0)}\cos(n\varphi+\varepsilon),\\ &w^{(0)}=w_n^{(0)}\cos(n\varphi+\varepsilon),\ N_1=N_{1n}\cos(n\varphi+\varepsilon),\ U=U_n\cos(n\varphi+\varepsilon),\\ &\qquad\qquad N_2=N_{2n}\sin(n\varphi+\varepsilon),\ V=V_n\sin(n\varphi+\varepsilon). \end{aligned}\right\} \tag{20}$$

The components of the external force- and couple-resultant are represented as follows:

$$\left.\begin{aligned} X=X_n\cos(n\varphi+\varepsilon),\ Y=Y_n\sin(n\varphi+\varepsilon),\ Z=Z_n\cos(n\varphi+\varepsilon),\\ \mathfrak{L}=\mathfrak{L}_n\sin(n\varphi+\varepsilon),\ \mathfrak{M}=\mathfrak{M}_n\cos(n\varphi+\varepsilon), \end{aligned}\right\} \tag{21}$$

* Ψ_1 and Ψ_2 denote the certain functions of X, Y, Z, $\mathfrak{L}$, $\mathfrak{M}$.

Substituting (20) and (21) in (14′), (16), (17), (18) and (19), we obtain *:

$$\left.\begin{aligned} N_{1n} &= \frac{D(1-\sigma)}{a^2(\nu^2+1)}\,\frac{d}{d\theta}[L_n(U_n)+U_n]-\frac{D}{a^2}\,\frac{dU_n}{d\theta}+ \\ &\quad +\frac{D(1-\sigma)}{a^2(\nu^2+1)}\,\Psi_{1n}+\mathfrak{M}_n, \\ N_{2n} &= -\frac{D(1-\sigma)}{a^2(\nu^2+1)}\,\frac{n}{\sin\theta}[L_n(U_n)+U_n]+ \\ &\quad +\frac{D}{a^2}\,\frac{n}{\sin\theta}\,U_n+\frac{D(1-\sigma)}{a^2(\nu^2+1)}\,\Psi_{2n}-\mathfrak{L}_n; \end{aligned}\right\} \tag{22}$$

$$L_n(\ldots)=\frac{d^2(\ldots)}{d\theta^2}+\operatorname{ctg}\theta\,\frac{d(\ldots)}{d\theta}+\left(1-\frac{n^2}{\sin^2\theta}\right)\cdot(\ldots); \tag{23}$$

$$\left.\begin{aligned} \Theta_n &= -\frac{1}{\nu^2+1}\,\frac{d}{d\theta}[L_n(U_n)+U_n]-\frac{1}{2}\,\frac{n}{\sin\theta}\,V_n-\frac{1}{\nu^2+1}\,\Psi_{1n}, \\ \Phi_n &= -\frac{1}{\nu^2+1}\,\frac{n}{\sin\theta}[L_n(U_n)+U_n]-\frac{1}{2}\,\frac{dV_n}{d\theta}+\frac{1}{\nu^2+1}\,\Psi_{2n}, \end{aligned}\right\} \tag{24}$$

$$L_nL_n(U_n)+\nu^2\cdot U_n+\left[l(\Psi_{1n})-\frac{n}{\sin\theta}\,\Psi_{2n}\right]=0; \tag{25}$$

$$L_n(V_n)+V_n-\frac{2}{\nu^2+1}\left[\frac{n}{\sin\theta}\,\Psi_{1n}+l(\Psi_{2n})\right]=0. \tag{26}$$

The corresponding homogeneous equations will be:

$$L_nL_n(U_n)+\nu^2\cdot U_n=0, \tag{27}$$

$$L_n(V_n)+V_n=0. \tag{28}$$

The first of these fall into two equations

$$L_n(U_n)+i\nu U_n=0, \tag{29}$$

$$L_n(U_n)-i\nu U_n=0. \tag{30}$$

The solutions of (29) will be obtained in hypergeometrical series $\left(\text{for } \theta<\frac{\pi}{2}\right)$:

$$\left.\begin{aligned} U_n^{(1)} &= \sin^n\theta\cdot F(a_n,\ b_n,\ c_n;\ \sin^2\theta), \\ U_n^{(2)} &= \sin^n\theta\cdot[F_1(a_n,\ b_n,\ c_n;\ \sin^2\theta)+ \\ &\quad +F(a_n,\ b_n,\ c_n;\ \sin^2\theta)\ln\sin^2\theta], \end{aligned}\right\} \tag{31}$$

where

$$a_n=\frac{2n+1}{4}+\frac{\sqrt{5+4i\nu}}{4},\quad b_n=\frac{2n+1}{4}-\frac{\sqrt{5+4i\nu}}{4},\quad c_n=n+1. \tag{31′}$$

Similarly, the solutions of (30) will be $\left(\text{for } \theta<\frac{\pi}{2}\right)$:

$$\left.\begin{aligned} \overline{U}_n^{(1)} &= \sin^n\theta\cdot F(\overline{a}_n,\ \overline{b}_n,\ c_n;\ \sin^2\theta), \\ \overline{U}_n^{(2)} &= \sin^n\theta\cdot[F_1(\overline{a}_n,\ \overline{b}_n,\ c_n;\ \sin^2\theta)+F(\overline{a}_n,\ \overline{b}_n,\ c_n;\ \sin^2\theta)\ln\sin^2\theta], \end{aligned}\right\} \tag{32}$$

* For $n=0$ and $\varepsilon=0$ the theory set forth below reduces to the theory of symmetrically loaded spherical shell as suggested by Meissner [2].

where

$$\overline{a}_n = \frac{2n+1}{4} + \frac{\sqrt{5-4i\nu}}{4}, \quad \overline{b}_n = \frac{2n+1}{4} - \frac{\sqrt{5-4i\nu}}{4}. \qquad (32')$$

If complex solutions of (31) and (32) be divided into real and imaginary parts

$$U_n^{(1)} = I_{1n} + i \cdot I_{2n}, \; U_n^{(2)} = I_{3n} + i \cdot I_{4n},$$
$$\overline{U}_n^{(1)} = I_{1n} - i \cdot I_{2n}, \; \overline{U}_n^{(2)} = I_{3n} - i \cdot I_{4n},$$

it is easy to obtain as a solution of (25):

$$U_n = \alpha_{1n} \cdot I_{1n} + \alpha_{2n} \cdot I_{2n} + \alpha_{3n} \cdot I_{3n} + \alpha_{4n} \cdot I_{4n} + \overline{U}_n, \qquad (33)$$

where $\overline{U}_n$ is a particular solution of (25) while α_{1n}, α_{2n}, α_{3n}, α_{4n} are arbitrary constants.

The solution of (26) is obtained in the following form:

$$\left.\begin{aligned} V_0 &= \beta_{10} \cdot \cos\theta + \beta_{20}\left(1 + \cos\theta \cdot \ln \operatorname{tg}\frac{\theta}{2}\right) + \overline{V}_0 && (n=0), \\ V_1 &= \beta_{11} \cdot \sin\theta + \beta_{21}\left(-\operatorname{ctg}\theta + \sin\theta \cdot \ln \operatorname{tg}\frac{\theta}{2}\right) + \overline{V}_1 && (n=1), \\ V_n &= \beta_{1n} \cdot \operatorname{tg}^n\frac{\theta}{2}(n + \cos\theta) + \beta_{2n} \cdot \operatorname{ctg}^n\frac{\theta}{2}(n - \cos\theta) + \overline{V}_n && (n>1), \end{aligned}\right\} \qquad (34)$$

where $\overline{V}_n$ is a special solution of (26), while β_{1n} and β_{2n} are arbitrary constants.

If (33) and (34) are substituted in (22) and (24), we obtain N_{1n}, N_{2n}, Θ_n, Φ_n after which $u_n^{(0)}$, $\varphi_n^{(0)}$, $w_n^{(0)}$ are easily found, the number of arbitrary constants being sufficient to satisfy any boundary conditions, from which a_{1n}, ... are directly determined. (The parallels $\theta = \theta_0$ and $\theta = \theta_1$ are the boundaries).

If the shell is closed at the apex, α_{3n}, α_{4n}, β_{2n} should be put equal to zero.

If in formule (20) and (21) we put $\varepsilon = \frac{\pi}{2}$ and let $\frac{n\pi}{\alpha}$ be substituted for n, then, analogously to what has been set forth above, it is possible to obtain the solution for the section of the sphere bounded by two parallels ($\theta = \theta_0$ and $\theta = \theta_1$), on which any conditions are satisfied, and two meridians ($\varphi = 0$ and $\varphi = \alpha$) along which the shell is freely supported.

V. Stekloff Mathematical Institute,
Academy of Sciences of the USSR,
Moscow.

Received
26. V. 1937.

REFERENCES

[1] A. Love. A Treatise of the Maheimatical Theory of Elasticity (1927). [2] E. Meissner, Phys. ZS., 14 (1913).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

HYDROTECHNICS

DETERMINATION OF THE ELEMENTS OF PERCOLATION FLOW UNDER DAMS WITH THREE CUT-OFF WALLS, RESTING UPON PERMEABLE FOUNDATION OF FINITE DEPTH

By V. S. KOSLOV

(Communicated by N. N. Pavlovsky, Member of the Academy, 10. V. 1937)

The classical theory of the flow of ground water under hydraulic structures, developed by Academician N. N. Pavlovsky, and further study based on that theory, afford a possibility for the solution of fairly complicated problems in ground water hydromechanics.

One of the practically important and, at the same time, complicated problems is that of flow under dams provided with three lines of cut-

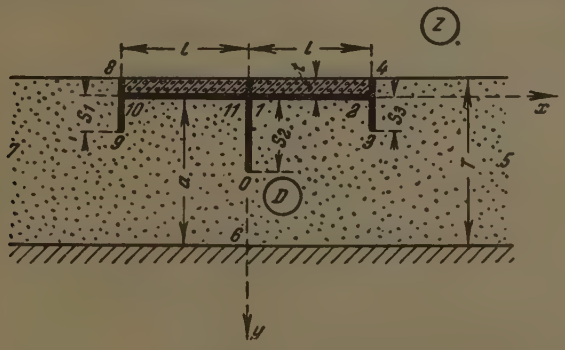


Fig. 1.

offs, the foundation being represented by a permeable layer of finite depth.

For a symmetrical scheme with three cut-off walls, the solution by the method of conformal transformation requires that the permeable region of the foundation be transformed on to a rectangular region, the hydrodynamical elements of which are known.

On account of the complicated shape of the profile, an intermediate transformation on to auxiliary half-plane becomes necessary.

The function transforming the region of the plane (z) on to the half-plane ζ, can be expressed, according to Schwarz-Kristoffel's formula, as follows.

$$z = A \int_0^{\zeta} \frac{\zeta (\sigma^2 - \zeta^2)\, d\zeta}{(\gamma^2 - \zeta^2) \sqrt{(\alpha^2 - \zeta^2)(\beta^2 - \zeta^2)(\alpha_0^2 - \zeta^2)}} + B.$$

The relation between the points of the planes z and ζ will be as follows:

$$z = s_2 i, \quad \zeta = 0; \quad z = 0, \quad \zeta = \sigma_0; \quad z = l, \quad \zeta = \alpha;$$
$$z = l + s_3 i, \quad \zeta = \sigma; \quad z = l - it, \quad \zeta = \beta; \quad z = \infty, \quad \zeta = \gamma;$$
$$z = (T - t)\, i, \quad \zeta = \infty.$$

On account of symmetry $B = 0$.

For the determination of five unknowns, it is necessary to compose five equations. But first of all should be simplified the function of transformation, determining the argument ζ from $\tau = \sqrt{\zeta^2 - \sigma_0^2}$, which gives

$$z = A_1 \int_0^{\tau} \frac{(\sigma_1^2 - \tau^2)\, d\tau}{(\gamma_1^2 - \tau^2) \sqrt{(\alpha_1^2 - \tau^2)(\beta_1^2 - \tau^2)}}.$$

This integral can be replaced by two other, more simple, integrals

$$z = A_1 (J_1 - J_2) =$$

$$= A_1 \Big[\int_0^{\tau} \frac{d\tau}{\sqrt{(\alpha_1^2 - \tau^2)(\beta_1^2 - \tau^2)}} - (\gamma_1^2 - \sigma_1^2) \int_0^{\tau} \frac{d\tau}{(\gamma_1^2 - \tau^2) \sqrt{(\alpha_1^2 - \tau^2)(\beta_1^2 - \tau^2)}} \Big].$$

The integral J_1 can be easily transformed into

$$J_1 = \frac{1}{\beta_1} \operatorname{sn}^{-1}\Big(\frac{\tau}{\alpha_1},\ k\Big) = \frac{U}{\beta_1},$$

where the modulus is equal to $k = \frac{\alpha_1}{\beta_1}$.

The integral J_2 is transformed into an elliptical integral of the third kind

$$J_2 = \frac{\gamma_1^2 - \sigma_1^2}{\beta_1 \gamma_1^2} \Big[U + \frac{\operatorname{tn} a}{\operatorname{dn} a} \Pi(U, a) \Big];$$
$$n = -k^2 \operatorname{sn}^2 a = -\frac{\alpha_1^2}{\gamma_1^2}; \quad \operatorname{sn} a = \frac{\beta_1}{\gamma_1}.$$

We further find $\operatorname{cn} a$, $\operatorname{tn} a$, $\operatorname{dn} a$ and

$$J_2 = \frac{\gamma_1^2 - \sigma_1^2}{\beta_1 \gamma_1^2} \Big[U + \frac{\beta_1 \gamma_1}{\sqrt{(\gamma_1^2 - \beta_1^2)(\gamma_1^2 - \alpha_1^2)}} \Pi(U, a) \Big].$$

Knowing that $z = A_1 (J_1 - J_2)$ we obtain:

$$z = \frac{A_1 \sigma_1^2}{\beta_1 \gamma_1^2} [U - \delta \Pi(U, a)], \qquad (1)$$

where

$$\delta = \frac{\gamma_1 \beta_1 (\gamma_1^2 - \sigma_1^2)}{\sigma_1^2 \sqrt{(\gamma_1^2 - \beta_1^2)(\gamma_1^2 - \alpha_1^2)}}. \qquad (2)$$

Let us now compose the five equations necessary for the solution of our problem:

a) The first equation refers to the point 2; for that point we have

$$z=l,\ \zeta=\alpha,\ \tau=\alpha_1,\ \text{sn}\ U=1,\ U=K$$

and

$$z=\frac{A_1\sigma_1^2}{\beta_1\gamma_1^2}[K-\delta\Pi(K,\ a)]=l.$$

Reverting to θ functions and Jacobi's function $Z(a)$ we have:

$$\frac{A_1\sigma_1^2}{\beta_1\gamma_1^2}K[1-\delta Z(a)]=l. \tag{3}$$

b) The second equation refers to the point 4; for that point we have:

$$z=l-it,\ \zeta=\beta,\ \tau=\beta_1,\ \text{sn}\,U=\frac{\beta_1}{\alpha_1}=\frac{1}{k},\ U=K+iK'.$$

Solving for θ functions and separating the imaginary part, we have:

$$\frac{A_1\sigma_1^2}{\beta_1\gamma_1^2}\left\{K'[1-\delta Z(a)]-\frac{\delta\pi a}{2K}\right\}=-t,$$

or, by virtue of equation (3)

$$\frac{K'}{K}l-\frac{A_1\sigma_1^2\delta\pi a}{\beta_1\gamma_1^2\cdot 2K}=-t. \tag{4}$$

c) The third equation refers to the point 6; for that point we have:

$$\zeta=\infty,\ \tau=\infty,\ U=iK'.$$

Reverting to θ functions, and taking into account equation (3), we have:

$$\frac{K'}{K}l-\frac{A_1\sigma_1^2\pi\delta}{\beta_1\gamma_1^2\cdot 2K}(a-K)=T-t,$$

or, by virtue of equation (4):

$$\frac{A_1\sigma_1^2\delta\pi}{\beta_1\gamma_1^2\cdot 2K}=T. \tag{5}$$

Taking the depth T as the basic dimension and putting $T=\frac{\pi}{2}$ we obtain:

$$\frac{A_1\sigma_1^2}{\beta_1\gamma_1^2}=\frac{1}{\delta}. \tag{6}$$

Then our equations (3), and (4) can be written as follows

$$\frac{K}{\delta}[1-\delta Z(a)]=l, \tag{7}$$

$$\frac{K'}{K}l-\frac{\pi a}{2K}=-t. \tag{8}$$

d) The fourth equation refers to the point 3; for this we have:

$$\zeta=\sigma,\ \tau=\sigma,\ \text{sn}\ U=\frac{\sigma_1}{\alpha_1},\ \frac{1}{k}>\frac{\sigma_1}{\alpha_1}>1,$$

therefore

$$U=K+i\lambda,$$

noting also that $\operatorname{sn}[(K+i\lambda);\ k]=\frac{1}{\operatorname{dn}(\lambda;\ k')}$, we have $\operatorname{dn}(\lambda;\ k')\frac{\alpha_1}{\alpha_1}$. Having in view equation (6) and substituting the value of U into the equation of the function of transformation, we obtain:

$$z=\frac{U}{\delta}-\Pi(U,\ a).$$

Rewerting to the θ function and taking only two terms of its series, we write:

$$l+s_3 i=\frac{l}{K}(K+i\lambda)-\frac{1}{2}\ln\frac{1+2q\cos\frac{\pi}{K}a\operatorname{ch}\frac{\pi}{K}\lambda+2iq\sin\frac{\pi}{K}a\operatorname{sh}\frac{\pi}{K}\lambda}{1+2q\cos\frac{\pi}{K}a\operatorname{ch}\frac{\pi}{K}\lambda-2iq\sin\frac{\pi}{K}a\operatorname{sh}\frac{\pi}{K}\lambda}.$$

Using the rule for taking logarithms of complex numbers and separating the real and the imaginary parts, we can write the fourth equation in the following form:

$$s_3=\frac{\lambda l}{K}-\operatorname{arctg}\frac{2q\sin\frac{\pi}{K}a\operatorname{sh}\frac{\pi}{K}\lambda}{1-2q\cos\frac{\pi}{K}a\operatorname{ch}\frac{\pi}{K}\lambda}. \qquad (9)$$

e) The fifth equation refers to the point 0; for that point we have:

$$\zeta=0,\ \tau=\sigma_0 i,\ \operatorname{sn}U=\frac{\sigma_0 i}{\alpha_1}$$

and, as before:

$$s_2 i=\frac{il\lambda_1}{K}-\frac{1}{2}\ln\frac{1-2q\cos(a-\lambda_1 i)}{1-2q\cos(a+\lambda_1 i)};$$

separating the real and the imaginary parts, we have:

$$s_2=\frac{\lambda_1 l}{K}+\operatorname{arctg}\frac{2q\sin\frac{\pi}{K}a\operatorname{sh}\frac{\pi}{K}\lambda_1}{1-2q\cos\frac{\pi}{K}a\operatorname{ch}\frac{\pi}{K}\lambda_1}. \qquad (10)$$

It will be observed that putting $\alpha_1=1$ we have in the second equation (8) only two unknowns, β_1 and γ_1, in the first equation (7) β_1, γ_1

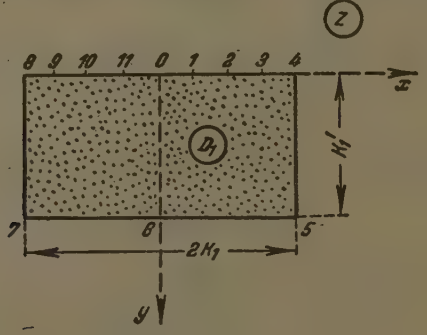


Fig. 2.

σ_1 are unknown, in the fourth equation (9) β_1, γ_1, σ_1 are unknown, in the fifth (10), only σ_0 is unknown, and in the third, A_1 is unknown. This property of our system of equations affords a convenient method

for ist solution. Let the variable $\frac{1}{\beta_1} = k$ run through a series of numerical values, for instance, 0.98; 0.96 etc. Determine, from the second (8) equation, a series of values of γ_1, for the given l and t, satisfying simultaneously with our value of β_1 the second equation of the system.

They will be determined as $\frac{\beta_1}{\gamma_1} = \text{sn}\,\frac{l}{T}K' = \text{sn}\ a$. Substitute these values into the first (7) equation, obtaining thus the value of σ_1 which satisfies, simultaneously with β_1 and γ_1, the first and the second equations of the system.

Having found σ_1, substitute expressions for β_1, γ_1 and σ into the fourth (9) equation, and obtain the values of the parameter s_3, satisfying this equation. Studying the equations we see that they do not depend upon s_2.

Furthermore, the value of σ_0 can be found, substituting into the fifth equation the values of β_1, γ_1, σ_1 and the required value of s_2.

As a result of the above described operations, we obtain the values of variables β_1, γ_1, σ_1 and σ_0 satisfying the four equations. The remaining equation (6) can serve for the determination of the coefficient A_1. Knowing β_1, γ_1, σ_1 and σ_0, the values of α, β, γ and σ can be found.

Having determined the function transforming the region to be studied on to the half-plane, we can find the affixes of the points of the rectangular model on to the same half-plane, knowing that $\zeta' = \pm 1$ at $Z = \pm K_1$ and $\zeta' = \pm \frac{1}{k_1}$ at $Z = \pm K_1 + iK_1'$.

Fractional-linear transformation gives

$$\zeta' = C\zeta,$$

where $C = \frac{1}{\beta}$; $k_1 = \frac{\beta}{\gamma}$.

After substitution and transposition, we obtain the function of transformation, as follows:

$$Z = \text{sn}^{-1}\left[\frac{f^{-1}(z)}{\beta}; \frac{\beta}{\gamma}\right]. \tag{11}$$

The functions representing the specific discharge q_r, the specific head h_r and the specific velocity v_r, can now be determined, from which the values of q, h and v are easily found, writing

$$q = \text{K}_\Phi H q_r, \quad h = h_2 + H h_r, \quad v = \text{K}_\Phi H v_r,$$

where K_Φ — the permeability coefficient,
H — acting head,
h_2 — the depth of water in the lower pool.

1. The equation for q_r is $q_r = \frac{K_1'}{2K_1}$ because $k_1^2 + k_1'^2 = 1$.

2. The equation for h_r is $h_r = \frac{1}{2}\left(1 - \frac{X}{K_1}\right)$.

For instance, for the point 0 (under the cut-off wall s_2) $X = 0$, $h_1 = \frac{1}{2}$; for the points 3 and 9 (under the cut-off walls s_1 and s_3), $X = \pm\,\text{sn}^{-1}\frac{\sigma}{\beta}$; for the points 1 and 11, $X = \pm\,\text{sn}^{-1}\frac{\sigma_0}{\beta}$; for the points 2 and 10, $X = \pm\,\text{sn}^{-1}\frac{\alpha}{\beta}$; for the points 8 and 4, $X = \pm K_1$.

3. The values of φ_r at the points where the water escapes into the lower pool can be determined by three ways:

a) $$\varphi_r = \frac{1}{2K_1} J\left(\frac{dZ}{dz}\right),$$

b) $$\varphi_r = \frac{1}{2K_1} J\left(\frac{\Delta X}{\Delta y}\right),$$

c) Considering the loss of head from the last cut-off to the lower pool and using, for the determination of φ_r, the equation derived for the case of flow around a single cut-off.

In order to simplify the practical application of the solution described in this paper, suitable diagrams have been compiled. The theoretical results were also checked by means of the electric analogy tests, using the apparatus as proposed and developed by Academician N. N. Pavlovsky.

The Water Resources Institute.
Ukrainian Academy of Sciences.
Kiev.

Received
27. V. 1937.

REFERENCES

[1] Н. Н. Павловский, Теория движений грунтовых вод под гидротехническими сооружениями и ее основные приложения (1922).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

MATHEMATISCHE PHYSIK

LÖSUNG DER ALLGEMEINEN AUFGABE DER DIFFRAKTION VON ELEKTROMAGNETISCHEN WELLEN

Von V. D. KUPRADZE

(Vorgelegt von I. M. Vinogradow, Mitglied der Akademie, d. 28. V. 1937)

Bisher wurden nur spezielle Fälle der Diffraktionsaufgabe behandelt und gelöst. Als Hindernisse, an denen die Erscheinungen der Diffraktion untersucht wurden, waren Kreis, Kugel oder einige andere einfachste Figuren angenommen, die eine effektvolle Anwendung der Fourierschen Methode durch Einführung von krummlinigen, dem gegebenen Fall entsprechenden Koordinaten zuliessen (1).

Es gehört A. Sommerfeld die elegante Theorie der Diffraktion am Spalt und linearem Winkel mit Hilfe von verzweigten Lösungen der Schwingungsgleichung im Riemannschen Raume (2). In meiner vorigen Arbeit (3) untersuchte ich und fand unter Anwendung der Integralgleichungsmethode die Lösung der Aufgabe der Diffraktion an einem Zylinder von beliebigem Durchschnitt. Diese Aufgabe kann man die ebene Diffraktionsaufgabe nennen.

In den eine Übersicht über den Gegenstand gebenden Abhandlungen von Laue und Epstein (4) wird darauf hingewiesen, dass «Die mathematische Schwierigkeiten der Beugungsaufgabe bringen es indessen mit sich, dass sie bis jetzt nur für eine sehr beschränkte Anzahl von speziellen Formen der beugenden Begrenzung durchgeführt werden konnte» und «Leider ist die Anwendbarkeit der Methode eine beschränkte und die Zahl der Fälle, deren Behandlung bisher gelungen ist, gering».

In der Sommerfeldschen Arbeit (2) wird dieser Umstand noch besonders unterstrichen. Die strenge Lösung der Aufgabe im allgemeinen Falle, d. h. der Aufgabe der Diffraktion an Hindernissen beliebiger Gestalt, stösst nach Sommerfeld an unüberwindliche mathematische Schwierigkeiten. Ähnliche Bewertungen dieser Aufgabe kann man in einer Reihe auch anderer speziellen Quellen begegnen (5).

In der vorliegenden Arbeit wird gezeigt, wie, unter Benutzung der Fredholmschen Theorie, die allgemeine Diffraktionsaufgabe gelöst werden kann.

Trifft die sich im dreidimensionalen Raume ausbreitende elektromagnetische Störung $\bar{\mathcal{E}}^*(x, y, z, t) = \bar{\mathcal{E}}(x, y, z)\, e^{-i\omega t}$ auf ihrem Wege das Hindernis V, das von einer genügend regulären Oberfläche einer willkürlichen Gestalt S begrenzt ist, so tritt eine Diffraktionserscheinung auf.

Das endgültige Feld wird durch zwei Vektore $H^*(x, y, z, t) = H(x, y, z)\, e^{-i\omega t}$ und $\overline{E}^*(x, y, z, t) = \overline{E}(x, y, z)\, e^{-i\omega t}$ die dem System der Maxwellschen Gleichungen:

$$\frac{\partial \overline{D}^*}{\partial t} = c \operatorname{rot} \overline{H}^* - \sigma \overline{E}^*, \quad \operatorname{div} \overline{D}^* = 0;$$

$$\frac{\partial \overline{B}^*}{\partial t} = -c \operatorname{rot} \overline{E}^*, \quad \operatorname{div} \overline{B}^* = 0 \quad (\varepsilon \overline{E}^* = \overline{D}^*, \ \mu \overline{H}^* = \overline{B}^*), \tag{1}$$

und den Randbedingungen:

$$E_{ti}^* = E_{ta}^*, \quad H_{ti}^* = H_{ta}^*,$$

genügen, charakterisiert. ε, μ, σ haben den Wert $\varepsilon_i, \mu_i, \sigma_i$ innerhalb V und entsprechend $\varepsilon_a, \mu_a, \sigma_a$ ausserhalb von V. Wenn, im besondern, das äussere Mittel den gewöhnlichen Luftraum darstellt (ein wichtiger Fall, z. B. in der drahtlosen Telegraphie), so ist

$$\varepsilon_a = \mu_a = 1, \quad \sigma_a = 0.$$

Für monochromatische Wellen $\overline{E}$ und $\overline{H}$ werden die Maxwellschen Gleichungen, wie bekannt [5], durch eine Gleichung, nämlich $\Delta \overline{u} + s^2(M)\, \overline{u} = 0$ ersetzt; hier ist $s^2(M)$ gleich $k_i^2 = \frac{\varepsilon_i H_i \omega^2 + i \mu_i \sigma_i \omega}{c^2}$, wenn M innerhalb V liegt, und ist gleich $k_a^2 = \frac{\varepsilon_a \mu_a \omega^2 + i \mu_a \sigma_a \omega}{c^2}$ falls M ausserhalb V ist; $\overline{u}$ ist gleich $\overline{E}$ oder $\overline{H}$ und die Randbedingungen gestalten sich zu:

$$E_{ti} = E_{ta}, \ \varepsilon_i E_{ni} = \varepsilon_a E_{na}; \quad H_u = H_{ta}, \ \mu_i H_{ni} = \mu_i H_{na}, \tag{2}$$

Die Indices i und a bezeichnen Grenzwerte der entsprechenden Grössen innerhalb bzw. ausserhalb. Indices t und n bezeichnen tangentielle bzw. normale Komponente.

Die Fläche S ist eine Quelle neuer Störungen und letztere stellen sich dar als eine Überlagerung sich ins Unendliche ausbreitender und dort abklingender Wellen (Strahlungsbedingung) [6], endlich muss die neue Störung im Falle des Nichtvorhandenseins von S in Null übergehen. Nun steht vor uns die Aufgabe, ein den Bedingungen:

a) $\Delta \overline{u} = s^2(M)\, \overline{u} = 0$;

b) $u_{ti} = u_{ta}, \ \varepsilon_i u_{ni} = \varepsilon_a u_{na}, \ \mu_i u_{ni} = \mu_a u_{na}$;

c) $(\overline{u} - \overline{\mathcal{E}})\, e^{-i\omega t}$ genügt der Strahlungsbedingung;

d) $(\overline{u} - \overline{\mathcal{E}}) = 0$, wenn $S \equiv 0$

genügendes $\overline{u}$ zu finden.

Allen diesen Bedingungen genügt die Lösung der folgenden belasteten vektoriellen Integralgleichung:

$$\overline{u} = \frac{k_i^2 - k_a^2}{4\pi} \int_V \overline{u} \frac{e^{ik_a r}}{r} d\nu + \frac{\lambda}{2\pi} \int_S \overline{n}_s u_{ni} \frac{\partial}{\partial n_s} \left(\frac{e^{ik_a r}}{r} \right) ds + \overline{\mathcal{E}}, \tag{3}$$

wenn nur die Lösung existiert.

Hier ist

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}; \quad \lambda = \frac{1}{2} \frac{\varepsilon_i - \varepsilon_a}{\varepsilon_a} \text{ oder } = \frac{1}{2} \frac{\mu_i - \mu_a}{\mu_a};$$

$\bar{\mathcal{E}}(x, y, z)$ genügt im ganzen Raume der Gleichung $\Delta\bar{\mathcal{E}} + k_a^2\bar{\mathcal{E}} = 0$, $\bar{n}$ das Einheitsvektor der äusseren Normale zu S ist. Überprüfen wir die Erfüllung der Bedingungen a), b), c), d).

Bedingung a). Es sei M ausserhalb V; nach Ausführung der Operation $\Delta + k_a^2$ in beiden Teilen von (3) erhalten wir $\Delta\bar{u} + k_a^2\bar{u} = 0$. Es sei M nun innerhalb V; nach Ausführung wieder derselben Operation $\Delta + k_a^2$ in beiden Teilen von (3) haben wir:

$$\Delta\bar{u} + k_a^2\bar{u} = -(k_i^2 - k_a^2)\bar{u}, \quad \text{d. i.} \quad \Delta\bar{u} + k_i^2\bar{u} = 0.$$

Bedingung b). Bilden wir in (3) die Differenz der Grenzwerte $\bar{u}$ ausserhalb und innerhalb und ziehen wir in Betracht die bekannte Eigenschaft der Unstetigkeit des Vektor-Potentials der doppelten Schicht auf S, so haben wir:

$$u_a(\sigma) - u_i(\sigma) = 2\lambda\, n_\sigma\, u_{ni}(\sigma).$$

Nach Projektion auf Tangente und Normale und Substitution des Wertes λ haben wir:

$$u_{ta} = u_{ti}, \quad \varepsilon_i u_{ni} = \varepsilon_a u_{na}; \quad \mu_i u_{ni} = \mu_a u_{na}.$$

Bedingung c). Durch Multiplikation aller Glieder der Gleichung (3) mit $e^{-i\omega t}$ finden wir:

$$\lim_{R\to\infty} R\left\{\frac{\partial}{\partial R}(\bar{u} - \bar{\mathcal{E}})\, e^{-i\omega t} + ik_a(\bar{u} - \bar{\mathcal{E}})e^{-i\omega t}\right\} = 0,$$

R ist der Halbmesser einer unendlich entfernten Kugel, mit dem Mittelpunkt im endlichen Teile des Raumes.

Bedingung d) folgt unmittelbar aus der Gestalt von (3).

Gleichung (3) kann nur eine einzige Lösung haben. Würde sie zwei Lösungen der geforderten Art haben, so würde ihre Differenz—genügend der homogenen Gleichung (3)—gleichzeitig mit der Normalableitung auf S stetig sein. So eine Funktion ist aber identisch gleich Null (7).

Gleichung (3) kann auf die gewöhnliche Fredholmsche Gleichung zurückgeführt werden, da Gleichung

$$\bar{u}^* = \frac{k_i^2 - k_a^2}{4\pi}\int_V u^*\frac{e^{ik_a r}}{r}\,dv$$

bei reellen ka nur die einzige Lösung $u^* = 0$ (7) hat.

Bezeichnen wir durch τ, τ' Punkte des Volumens, durch s, σ Punkte auf S,

$$\frac{k_i^2 - k_a^2}{4\pi}\frac{e^{ik_a r}}{r} = K_1(\tau, \tau'), \quad \frac{\lambda}{2\pi}\frac{\partial}{\partial n_s}\left(\frac{e^{ik_a r}}{r}\right) = K_2(\tau, \tau')$$

und schreiben nun (3) als

$$\bar{u}(\tau) = \int_V \bar{u}(\tau')\, K_1(\tau, \tau')\, d\tau' + \int_S \bar{n}_s\, u_{ni}\, K_2(\tau, s)\, ds + \bar{\mathcal{E}}(\tau). \tag{4}$$

Bezeichnet $\Gamma(\tau, \tau')$ die Resolvente des Kernes $K_1(\tau, \tau')$ so haben wir aus (4):

$$\bar{u}(\tau) = \bar{\mathcal{E}}(\tau) + \int_S \bar{n}_s u_{ni}\, K_2(s, \sigma)\, ds +$$

$$+ \int_V \Gamma(\tau, \tau')\left\{\bar{\mathcal{E}}(\tau') + \int_S \bar{n}_s\, u_{ni}(s)\, K_2(s, \tau')\right\} d\tau'. \tag{5}$$

Strebt τ zu (σ) auf S, so haben wir aus (5):

$$\bar{u}(\sigma) - \lambda \bar{n}_\sigma u_{ni}(\sigma) = \bar{\mathcal{E}}(\sigma) + \int_S \bar{n}_s u_{ni}(s) K_2(s,\sigma)\, ds +$$

$$+ \int_V \Gamma(\sigma,\tau')\, \mathcal{E}(\tau')\, d\tau' + \int_S \int_V \Gamma(\sigma,\tau')\, \bar{n}_s u_{ni} K_2(s,\tau')\, d\tau\, ds. \qquad (6)$$

Dies ist aber eine Fredholmsche Gleichung zweiter Art, die auf Grund des oben bewiesenen Satzes über die Einzigkeit für (3) dazu auch nur eine einzige Lösung hat. Finden wir aus (6) diese Lösung und substituieren sie in die rechte Seite der Gleichung (5), so kommen wir zur Lösung unseres Problems.

Nachdem der Existenzsatz bewiesen ist, ist es am zweckmässigsten um numerische Näherungswerte zu erhalten, die üblichen Approximationsmethoden zur Lösung der Integralgleichung (3) anzuwenden.

Mathematisches Institut
der Georgischen Filiale der
Akademie der Wissenschaften
der UdSSR.

Eingegangen
d. 28. V. 1937.

ZITIERTE LITERATUR

[1] Handbuch d. Physik, XX. [2] Frank u. Mieses, Die Diff. u. Integralgl. der Physik, Kap. (1936). [3] В. Купрадзе, Тр. Тбилисского мат. ин-та, 1 (1937). [4] Enc. d. Math. Wiss., V, 24; M. von Laue, Wellenpolitik; P. S. Epstein, Spezielle Beugungsprobleme. [5] Handbuch d. Physik, XX. Wolfson, Strenge Theorie d. Interferenz u. Beugung. [6] A. Sommerfeld, Jahresbericht d. D. Math. Ver., 21 (1912). [7] В. Купрадзе, ДАН, I, 2 (1934).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

PHYSIQUE

LA BANDE DE DISPERSION ET LE SKIN-EFFET DANS LE CHAMP SINUSOÏDAL ET DANS CELUI DE TRANSITION

Par V. K. ARKADIEV (W. Arkadiew), membre correspondant de l'Académie

Supposons que par suite de la viscosité magnétique une partie de l'intensité d'aimantation I dans le champ constant instanément créé apparaît immédiatement, tandis que l'autre croît exponentiellement (1). Dans le champ H au moment $t = 0$

$$I = \varkappa_1 H;$$

à $t > 0$

$$I = \varkappa_1 H + I',$$

et

$$I' = \varkappa_2 (1 - e^{-at}) H,$$

où $a = \frac{1}{\tau_u}$; τ_u est la constante du temps de la relaxation. A $t_{\to\infty}$

$$I' = I_2 = \varkappa_2 H$$

et

$$I = (\varkappa_1 + \varkappa_2) H.$$

1. L'induction magnétique $B = 4\pi I + H$, d'où

$$\mu = 1 + 4\pi\varkappa_1 + 4\pi\varkappa_2 (1 - e^{-at}). \qquad (1)$$

Si l'on porte en abscisses lg t, les variations de l'induction magnétique peuvent être présentées par la courbe $B = \mu H$; au début du processus

$$B = (1 + 4\pi\varkappa_1) H = \mu_1 H.$$

Pendant un temps assez long la courbe monte de $4\pi\varkappa_2 H$, B atteignant la valeur $B_\infty = (\mu_1 + 4\pi\varkappa_2) H = \mu_\infty H$. Au moment où $t = \tau_u$, l'élévation de la courbe n'atteint que 0.63 de l'élévation totale. A $t = \frac{\tau_u}{4}$ l'élévation fait 0.22; à $t = 2\tau_u$ elle fait 0.86. Le rapport de l'induction à t très grand (B_∞) et celle à t très petit, lorsque $B = \mu_1 H = B_1$ est

$$\frac{B_\infty}{B_1} = \frac{\mu_\infty}{\mu_1}.$$

2. Quand le champ à période T varie périodiquement, la composante conservative de la perméabilité μ sur l'échelle logarithmique des périodes varie symétriquement à l'égard de la période $T_u = \frac{2\pi}{a}$ selon la formule (2):

$$\mu = \mu_1 + 4\pi\varkappa_2 \frac{a^2}{a^2 - \omega^2}, \qquad (2)$$

où $\omega = \frac{2\pi}{T}$. En effet, à $T = T_u$ et $a = \omega$ l'élévation atteint 0.5; à $T = \frac{T_u}{2}$ et $\omega = 2a$ elle fait 0.2 et à $T = 2T_u$ et $\omega = \frac{a}{2}$ l'élévation est égale à $1 - 0.2$.

Désignons le rapport de l'amplitude de la composante conservative de l'induction B à très grandes périodes à celle à très petites périodes par $\frac{B_\infty}{B_1}$. Alors

$$\frac{B_\infty}{B_1} = \frac{\mu_\infty}{\mu_1}.$$

3. Examinons à présent le processus de la pénétration du champ constant dans le métal ferromagnétique à viscosité magnétique et à conductibilité électrique γ cgsm.

La couche spécifique du flux magnétique qui parvint à pénétrer dans le métal (3) est

$$\phi = \int_0^\infty B dz;$$

tant que t est très petit et l'aimantation visqueuse I' ne s'est pas encore montrée, la couche fait

$$\frac{H}{\pi}\sqrt{\frac{\mu_1 t}{\gamma}} = \phi_1.$$

Il résulte de la table 3 de Tikhonov (4), donnée pour le cas où $4\pi\varkappa_2 = \mu_1$, que ϕ diffère de la valeur ϕ_1 ci-dessus mentionnée moins que de 2% tant que $t < \frac{\tau_u}{10}$.

Avec cela la profondeur équivalente de la pénétration de l'induction dans le métal z_e, qui est en général égale à $\frac{\phi}{\mu H}$, est moins grande que

$$z_{e1} = \frac{1}{\pi}\sqrt{\frac{\tau_u}{10\gamma\mu_1}}.$$

Sur la fig. 1 est portée en abscisses $\sqrt{\frac{t}{\tau_u}}$.

Le diagramme fait voir que le processus avance par la droite inférieure jusqu'à l'abscisse 0.3, tandis qu'après 1.4 il passe à la droite supérieure, qui possède un coefficient angulaire plus élevé.

Il s'ensuit du diagramme, construit d'après les données de Tikhonov, qu'après $t = 2\tau_u$ l'aimantation visqueuse atteint un développement presque complet et se rapproche de $\varkappa_2 H$, tandis que la perméabilité s'approche de μ_∞. Alors la couche au lieu d'être

$$\phi_2 = \frac{H}{\pi}\sqrt{\frac{\mu_\infty t}{\gamma}}$$

est $\psi = \psi_2 - \psi'$, où ψ' est presque constant; approximativement

$$\psi' = 0.17H\sqrt{\frac{\mu_1 \tau_u}{4\pi\gamma}};$$

par conséquent

$$z_e > z_{e12} = \frac{1}{\pi}\sqrt{\frac{\tau_u}{\mu_1\gamma}} - \frac{\psi_1}{\mu_\infty H}.$$

Ayant des valeurs moyennes de t, notamment lorsque

$$\frac{\tau_u}{10} < t < 2\tau_n \text{ et } z_{e1} < z_e < z_{e12},$$

il faut calculer la couche entre ces limites d'après les tables et les courbes de Tikhonov. Le rapport des limites indiquées z_e fait $z_{e12} : z_{e1} = \sqrt{10} - 0.17\sqrt{\frac{5}{8}}\pi \approx 3$. Le rapport des temps correspondants fait 20.

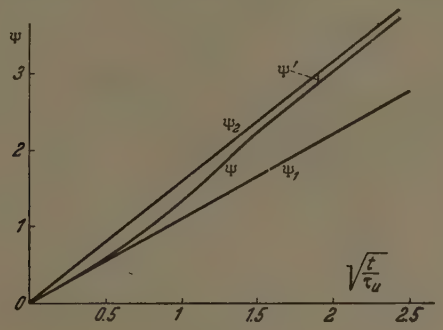


Fig. 1.

Avec des t moyens le processus passe par le stade de transition, analogue à la partie centrale des deux premières courbes, qui présentent la région de dispersion. Si nous prenons le rapport $\frac{\psi}{\psi_1}$, nous obtenons une courbe analogue aux deux premières; t étant grand

$$\frac{\psi}{\psi_1} = \sqrt{\frac{\mu_\infty}{\mu_1}} - 0.17\sqrt{\frac{\pi\tau_u}{4t}}.$$

Avec $t \to \infty$ ce rapport a pour limite $\sqrt{\frac{\mu_\infty}{\mu_1}}$.

Le rapport $\left(\frac{\psi}{\psi_1}\right)^2$ peut être considéré comme mesure d'un accroissement progressif de la perméabilité effective, déterminant le processus. La valeur

$$\mu_0 = \mu_1\left(\frac{\psi}{\psi_1}\right)^2 \qquad (3)$$

représente la perméabilité apparente «μ», qu'on détermine d'après le processus de la propagation du champ dans le métal à viscosité magnétique, en supposant que la viscosité n'existe guère, c'est-à-dire, comme si

$$\psi = \frac{H}{\pi}\sqrt{\frac{\mu_0 t}{\gamma}}.$$

La courbe *1* des fig. 2 et 3 présente la fonction $1 - e^{-\frac{t}{\tau_u}}$, et la courbe *2* celle de $\frac{a^2}{a^2+\omega^2}$, voir (1) et (2); la courbe *3* présente le rapport $\frac{\mu_v - \mu_1}{\mu_1}$. Ce dernier caractérise l'accroissement de la perméabilité apparente μ_v et sa

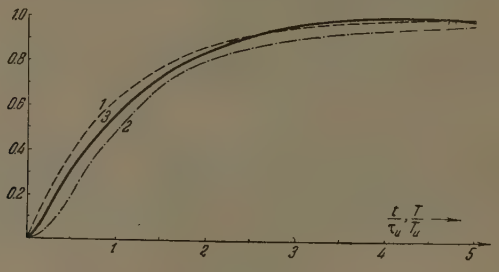


Fig. 2.

variation de μ_1 jusqu'à $\mu_1 + 4\pi\varkappa_2$. Sur la fig. 2 c'est $\frac{t}{\tau_u}$ qui sert d'axe des abscisses pour les courbes *1* et *3*, et $\frac{T}{T_u}$ pour la courbe *2* — tandis que sur la fig. 3 ce sont leurs logarithmes.

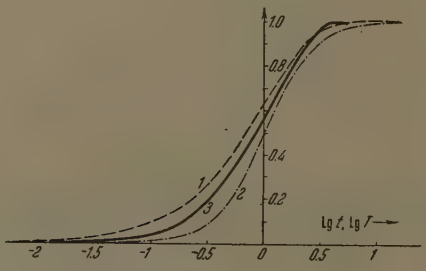


Fig. 3.

4. Dans le cas du champ alternatif sinusoïdal (5) l'amplitude de la couche spécifique du flux est

$$\Phi_0 = \frac{H_0}{2\pi}\sqrt{\frac{r_m T}{2\gamma}},$$

où $r_m = \sqrt{\mu^2 + \rho'^2}$ et

$$\rho = \frac{4\pi\varkappa_2 a}{a^2+\omega^2}.$$

Les limites des valeurs r_m sont μ_1 et μ_∞. La limite du rapport

$$\frac{\Phi_0}{\Phi_{01}} = \sqrt{\frac{r_m}{\mu_1}}$$

à $T \gg 2\pi\tau_u$ est 1, et à $T \gg 2\pi\tau_u$:

$$\frac{\Phi_0}{\Phi_{01}} = \sqrt{\frac{\mu_\infty}{\mu_1}}.$$

La construction des courbes démontre([5a]) que le cours r_m dépendant du lg T est presque le même que μ dans le champ périodique [v. (2)]. C'est pourquoi la bande de dispersion diffère ici bien peu de la courbe μ.

De ce que nous avons établi il en résulte encore le fait suivant. Le champ constant créé pénètre dans le métal avec une vitesse décroissante. En effet, la viscosité étant absente, la profondeur équivalente z_e de la pénétration de l'induction, c'est-à-dire la profondeur où l'induction fait 0.425 de l'induction à la couche superficielle, est ([6])

$$z_e = \frac{1}{\pi}\sqrt{\frac{t}{\mu\gamma}};$$

d'où la vitesse de la propagation de cette valeur de l'induction (0.425) est

$$\frac{dz_e}{dt} = \frac{1}{2\pi\sqrt{\mu\gamma t}} = \frac{1}{2\pi^2\mu\gamma z_e}. \qquad (4)$$

Au début du processus, c'est-à-dire à la surface même du corps, la vitesse d'apparition du champ magnétique H dans le point donné est très grande. A de grandes profondeurs z les impulsions affluentes du champ et de l'induction croissent plus lentement. Par conséquent, an commencement de la propagation l'induction magnétique diffère peu de sa valeur initiale $\mu_1 H$. Par contre, àde grandes profondeurs, où le champ n'apparaît que fort lentement, l'accroissement de l'induction ne doit être ralenti ni par la viscosité, ni par l'inertie, et elle atteint, avec le champ, la valeur statique correspondante $\mu_\infty H$. Un phénomène analogue se fait voir quand on fait passer la substance étudiée à travers le spectre à différentes périodes d'oscillations. Les courbes μ et ϕ peuvent être considérées comme modes de représentations de la même propriété spectrale de la substance. Dans les deux cas apparaît la bande spectrale de dispersion. Toutefois il y a lieu de noter que généralement la substance passe à travers le spectre avec une vitesse uniforme, tandis que dans le cas de la courbe ϕ a lieu, au moment du passage par les périodes de la bande, une réduction extrême de la vitesse d'accroissement du champ à la suite de l'augmentation de μ dans la formule (4), produite par cette bande.

Les études ultérieures peuvent porter sur le processus de la propagation de l'induction dans le métal conducteur à champ constant, lorsque les centres magnétiques de la substance possèdent non seulement la viscosité, mais aussi l'inertie magnétique. En ce cas dans la solution des équations initiales faite par A. N. Tikhonov l'expression

$$\varkappa_2 H - I' = \tau_u \frac{dI'}{dt}$$

doit être remplacée ([7]) par

$$\varkappa_2 H - I' = \tau_u \frac{dI'}{dt} + \tau_0^2 \frac{d^2 I'}{dt^2},$$

où τ_0 est une nouvelle constante.

Il est à noter que la couche spécifique du flux ϕ permet de calculer le flux magnétique Φ, passant à la surface du cylindre le long de son axe, s'il est orienté dans la direction H. Dans ce cas

$$\Phi = 2\pi r\phi,$$

où r est le rayon du cylindre. L'expression est juste ([8]) tant que $q = \frac{r}{2z_{e^2}} > 4.5$.

Ainsi, en régistrant à l'aide d'un oscillographe les premiers moments de l' apparition du flux Φ dans les champs faibles du cylindre ferromagnétique, c'est-à-dire à μ ne dépendant pas du champ, on obtient facilement une courbe, qui permet de calculer le spectre magnétique de la matière en question.

En conclusion notons qu'en calculant la résistance magnétique du cylindre, on trouve les traits d'analogie communs aux calculations de l'influence du skin-effet dans les champs de transition et dans les champs périodiques. En absence d'hystérèse et de la viscosité nous avons l'expression de la résistance conservative du cylindre pour le champ sinusoïdal

$$R'_m = AR_m,$$

où $R_m = \frac{l}{\mu S}$ est sa résistance magnétique en présence d'un champ magnétique constant.

Lorsque le paramètre

$$\varkappa = \pi r \sqrt{\frac{\mu\gamma}{T}} > 1.5,$$

on a

$$A = \varkappa + 0.28.$$

La résistance du cylindre au moment t après l'apparition d'un champ constant a été présentée par nous sous une forme analogue [8]:

$$R'_m = aR_m$$

où a est la fonction de

$$q = \pi r \sqrt{\frac{\mu\gamma}{4t}};$$

dans le cas de $q > 1.6$, on a

$$a = q + 0.20.$$

L'analogie de ces deux schémas du calcul est évidente. Notons que dans le cas du champ sinusoïdal le quart de la période T joue le rôle du temps t.

Manuscrit reçu
le 26. V. 1937.

LITTÉRATURE CITÉE

[1] W. Arkadiew, Z. S. f. Ph., 72, 116 (1931). В. К. Аркадьев, Электромагнитные процессы в металлах, II, p. 14 (1936); ЖЭТФ, 7, № 1, 131 (1937). [2] L. c., p. 17 et 266. [3] L. c., p. 35. [4] А. Тихонов, ЖЭТФ, 7, № 1, 138 (1937). [5] L. c., p. 45. [5a] L. c., p. 268. [6] L. c., 35. [7] L. c., p. 292. [8] L. c., p. 66.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

PHYSICS

TORCH ION COUNTER

By A. PROKOFJEV

(Communicated by S. I. Wawilow, Member of the Academy, 7. V. 1937)

A new type of ion counter is described in this paper. The well-known phenomenon of torch-like effluence, as described by Prof. S. I. Zilitinkevič ([1]), may be made use of as an ion counter. This phenomenon is as follows: when a powerful enough short-wave generator is at work, a flame-like flare, not unlike in appearance the flame of a gas torch, may be observed on the circuit conductor carrying a high enough frequency potential. The flame of the torch is a steady one so long as the operating conditions of the generator remain unchanged, a change in the latter causing a corresponding change in the torch. If an alternating voltage is supplied to the anode of the generator lamp, the torch burns only a fraction of the positive half-period. It is therefore not difficult to quench a torch formed on electrode A by placing it in the constant electric field between the two plates of a flat condenser (Fig. 1). The magnitude of the field may be controlled at will by means of a constant battery E. The intensity value of the constant field required to quench the torch depends on the intensity value of the high-frequency field at the electrode A, the former being the greater, the greater the high-frequency field. In order to quench a torch of about 200-watt initial power formed on a brass electrode A ending in a sphere of a radius of 2 mm, a constant potential difference should be applied to the plates of a flat 18 000-volt condenser C, the distance between the discs of the condenser being 20 cm and their diameter 40 cm. In watching the behaviour of the whole arrangement after the field has reached an intensity sufficient to quench the torch formed, one may observe flashes of a peculiar nature appearing from time to time on electrode A, the number of flashes per minute depending on the electric condition of the air around electrode A. The number of flashes is increased if the space around electrode A is exposed to the radiation of a radio-active substance, to the action of X-rays, or ultra-violet light (open voltaic

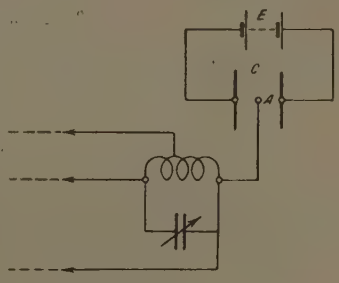


Fig. 1.

arc). It has also been observed that when the heat current produced by a spirit lamp passes in the vicinity of electrode A, the number of flashes is considerably increased. These observations show that the torch-like effluence may, in the conditions described above, be used as an ion counter. It being interesting to compare the performance of the new counter with that of a Geiger-Müller counter of the usual type, the following experiments were made. After satisfying ourselves as to the steadiness of the work of the torch counter when it is not exposed to the radiation of the above agents, we made a record of the number of spontaneous flashes per minute. Then a 0.078-mg

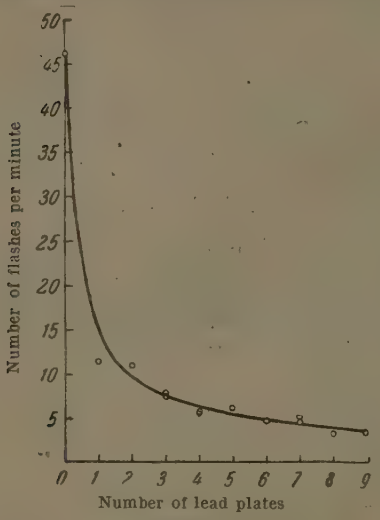


Fig. 2.

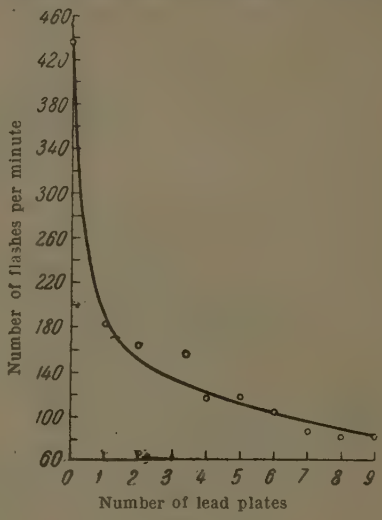


Fig. 3.

radium preparation was placed at a distance of 62 cm from electrode A, the number of flashes per minute being again noted down. Subsequently, lead plates, each 0.2 cm thick, were placed between the bulb and the electrode. The results obtained are shown in Table 1 and in Fig. 2.

Table 1

No of measurement	Average number of spontaneous flashes per minute	Number of lead plates	Average number of all flashes per minute	Average number per minute of flashes due to radio-active substance
1	3	0	—	—
2	3	0	49.1	46.1
3	3	1	14.5	11.5
4	3	2	13.8	10.8
5	3	3	11.0	8.0
6	3	4	8.7	5.7
7	3	5	9.3	6.3
8	3	6	8.0	5.0
9	3	7	8.5	5.5
10	3	8	6.5	3.5
11	3	9	6.7	3.7

Similar measurements were made with a Geiger-Müller counter, the length of the latter being 8 cm, and its diameter 2.5 cm. The counter was enclosed in a case of 0.1 mm brass foil. The bulb was, as before, 62 cm distant from the counter. Lead plates, 0.2 cm thick, were placed between the bulb and the counter. The results obtained are shown in Table 2 and in Fig. 3.

Table 2

No of measurement	Average number of spontaneous throws per minute	Number of lead plates	Average number of all throws per minute	Average number per minute of throws due to radio-active substance
1	71	0	—	—
2	71	0	537	436
3	71	1	255	182
4	71	2	237	164
5	71	3	230	157
6	71	4	189	116
7	71	5	191	118
8	71	6	181	104
9	71	7	158	85
10	71	8	155	82
11	71	9	155	82

The Geiger-Müller counter curve is somewhat less steep. This may be due to the relay error (the number of throws here being very great at beginning of the curve); this question, however, requires further investigation. A comparison of the sensitivity of the torch counter with that of a Geiger-Müller counter shows the latter to be higher. This may be due to the greater working surface of the Geiger-Müller counter, and also to the fact that the torch counter works only a fraction of the positive half-period.

In conclusion, some of the advantages of the torch counter may be mentioned. It can count not only fast, but also slow ions, as may be seen from the spirit-lamp experiment. All the characteristics of the counter, such as the influence of the magnitude of the constant field on the number of flashes, the influence of the temperature of the electrode, the influence of the sign of the charge, and others, will be described in subsequent articles.

I take this opportunity of conveying my thanks to I. V. Kurčatov, S. I. Zilitinkevič, and V. V. Basilevič for the interest they took in my work.

Institute of Experimental Meteorology.
Leningrad.

Received
7. V. 1937.

REFERENCES

[1] С. И. Зилитинкевич, Телеграфия и телефония без проводов, IX, № 6 (51), декабрь (1928).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

BIOPHYSIQUE

ÉTUDE THÉORIQUE DE L'INFLUENCE DE L'ASCENSION DES MONTAGNES ET DE L'INFLUENCE DES VENTS SUR L'ADAPTATION VISUELLE PÉRIPHÉRIQUE

Par P. LASAREFF, de l'Académie

1. Dans ma note précédente (1) j'ai démontré que les changements ΔE_0 de la sensibilité maximum de l'œil E_0 au cours de la vision périphérique dépendent de la saison $[\Delta E_0(s)]$, de l'heure du jour $[\Delta E_0(h)]$, de la latitude φ du lieu d'observation $[\Delta E_0(\varphi)]$.

J'ai développé une théorie thermique de ces changements en admettant que les variations de E_0 dépendent des variations de la température ϑ du milieu ambiant. En étudiant les variations de E_0 pendant un an et les changements de la température moyenne ϑ de chaque mois, on peut trouver une corrélation de ϑ et E_0, ce qui permet d'expliquer théoriquement les variations de E_0 pendant un jour et le long du méridian.

Pour donner l'allure théorique des courbes de $\Delta E_0(h)$, ΔE (φ) nous admettons que ces variations dépendent également des changements de la température. On peut faire encore une hypothèse que l'influence de la température ϑ joue un grand rôle pendant les ascensions des montagnes et sous l'action des vents.

Dans la note présente je donne une théorie des changements de la sensibilité pendant les ascensions des montagnes et des changements produits par l'action des vents.

2. Les observations de T. N. Tschigevskaia (Čiževskaia) (2) ont démontré que la valeur de E_0 devient plus grande sur les montagnes. Tschigevskaia a constaté cet effet en mesurant E_0 à Jalta (hauteur 70 m) et au sommet de Ai-Petri (hauteur 1180 m); elle a trouvé un agrandissement considérable de E_0 sur les montagnes.

Les températures ϑ à Jalta et à Ai-Petri étaient différentes (à Jalta $\vartheta = 27°$ et à Ai-Petri 7°). Nous devons donc admettre l'existence de l'effet simultané de la température ϑ et de la pression p, l'effet de la température étant prépondérant. Pour pouvoir observer l'effet de la pression avec une température invariable, Tschigevskaia* a entrepris des observations de E_0 avec une installation munie d'appareils permettant de changer la pression p de l'air dans une chambre spéciale (barochambre).

---

* Les résultats des expériences de T. N. Tschigevskaia (Čizevskaia) ont été communiqués dans le Colloquim biophysique de l'Institut de médecine expérimentale (Moscou).

Ces expériences éxécutées avec des pressions correspondantes à la hauteur maximum de 4000 m duraient 5 heures. Elles ont confirmé en général les résultats reçus dans les recherches de Višnievski et Cyrline (3) dont le but était de constater les variations rapides de E_0 en avion.

Pendant la première observation avec la pression correspondante à la hauteur de 1200 m Tschigevskaia a observé une petite diminution de E_0.

Pendant la seconde expérience avec la même pression, éxecutée 3 jours après la première, elle a observé un petit agrandissement de E_0 qui s'augmentait avec la répétiton des expériences. Pendant la troisième expérience le changement de E_0 est plus grand que pendant la seconde et ainsi de suite.

Nous voyons donc que les observations de E_0 au cours de l'ascension des montagnes et les mesures de E_0 faites avec des pressions correspondantes aux mêmes hauteurs sont en discordance.

Nous avons déjà indiqué que la température au niveau de la mer et sur les montagnes dans les expériences de Tschigevskaia était différente. Nous calculons d'abord les variations de E_0 produites par la température sur les montagnes en négligeant l'influence de la pression.

La température à Jalta était 27° et à Ai-Petri 7°. En se basant sur la correlation de ϑ et E_0 obtenue dans mon travail antérieur (4), on peut trouver la valeur de E_0 à Jalta et à Ai-Petri. Dans le tableau nous avons ces données

ϑ	27°	7°
E_0	0.61	0.99
Lieu d'observation	Jalta	Ai-Petri

D'après ces valeurs on peut calculer le rapport de E_0 à Ai-Petri et E_0 à Jalta, si l'on néglige l'influence de la pression.

Ce rapport E_0(A,J) est égal à 1.62.

E_0(A,J) a été mesuré directement par Tschigevskaia pendant son ascension sur Ai-Petri; elle l'a trouvé égal à 2.00. Nous voyons donc qu'avec une erreur de 20% on peut calculer E_0 sur les montagnes en n'admettant que l'influence de la température.

En se rappelant l'influence de la pression constatée par Tschigevskaia pendant les expériences dans la chambre à pression diminuée, nous pouvons obtenir une coïncidence encore plus parfaite. Nous avons indiqué que E_0 pendant la première expérience montre une petite diminution, laquelle se transforme dans un agrandissement pendant la deuxième expérience.

On peut admettre qu'un séjour prolongé dans l'athmosphère raréfiée produit d'abord une diminution et puis un petit agrandissement de E_0, dépendant de la pression. Cette loi est confirmée par les observations de Tschigevskaia à Ai-Petri.

On peut admettre ensuite que la variation de E_0, dépendant de la pression, au cours des expériences éxécutées sur Ai-Petri, correspond à la seconde expérience dans la barochambre.

Avec ces hypothèses on doit affirmer que E_0 sur les montagnes en été montre une augmentation dépendant de la variation de la température et égale à 1.62 et une augmentation sous l'influence de la pression qui est égale à 1.2.

Nous avons par conséquent

$$E_0(\mathrm{A,J}) = 1.62 \cdot 1.2 = 1.94.$$

Le résultat des calculs théoriques coïncide très bien avec la valeur E_0(A,J) mesurée par Tschigevskaia et égale à 2.00.

3. L'effet produit sur la sensibilité E_0 par les vents doit dépendre principalement de l'échauffement ou du refroidissement du corps de l'homme et, par conséquent, doit être réduit à l'influence de la température et de l'humidité du milieu ambiant.

Le refroidissement et l'échauffement produits par les vents dépendent de la vitesse du vent, de la température et de l'humidité de l'air en mouvement. On peut admettre qu'il existe une température définie δ_0 de l'air avec laquelle le vent ne produit pas les changements de la température du corps et n'a aucune influence sur la valeur E_0. Les températures plus grandes ou plus petites provoquent l'échauffement oule refroidissement du corps et conduisent à la diminution de E_0.

Dans mon laboratoire, sous ma direction, Tschigevskaia et Speranski ont éxécuté des expériences* sur l'influence des mouvements de l'air sur la sensibilité visuelle périphérique. L'air en mouvement avait toujours la même vitesse et les températures différentes. On a constaté que la variation continue de la température de l'air change la sensibilité E_0, qui a un maximum à la température égale à 21° environ.

Ces données permettent de conclure que la température de l'air et son humidité sont les causes principales de l'influence des vents sur la sensibilité.

Institut de Géographie de l'Académie
des Sciences de l'URSS.Section géophysique.
Moscou.
Institut de médecine expérimentale.
Sverdlovsk.

Manuscrit reçu
le 26. V. 1937.

LITTÉRATURE CITÉE

[1] P. P. Lasareff, C. R. Acad. Sci. URSS, XIV, 275 (1937). [2] P. P. Lasareff et T. N. Čiževskaia, C. R. Acad. Sci. URSS, XIII, 309 (1937). [3] Н. А. Вишневский и Б. Цирлин, Труды Первой конференции по физиологической оптике, 158 (1936). [4] P. P. Lasareff, loc. cit., 276.

* Le travail de P. P. Lasareff, T. N. Tschigevskaia (Čizevskaia) et A. P. Spéranski, malgré le nombre considérable des observations n'est pas encore fini.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

GEOPHYSIK

ÜBER DIE BESTIMMUNG DER GEOIDFIGUR AUS DEN ANOMALIEN DES HORIZONTALGRADIENTEN DER ERDSCHWERE

Von N. MIHAL

(Vorgelegt von S. I. Wawilow, Mitglied d. Akademie, d. 7. V. 1937)

Wir wollen von der Formel von Stokes (1)

$$N = \frac{R}{4\pi\gamma} \int \Delta g S \psi \, d\sigma \tag{1}$$

ausgehen; hier bedeuten N — den Abstand des Geoids vom Niveauellipsoid, $d\sigma$ — das Flächenelement der Einheitskugel, Δg — die Schwereanomalie, bezogen auf das Flächenelement $d\sigma$:

$$S(\psi) = \frac{1}{\sin\frac{\psi}{2}} - 6\sin\frac{\psi}{2} - 3\cos\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) + 1 - 5\cos\psi,$$

ψ — den sphärischen Abstand von $d\sigma$ vom Punkt P, in welchem N berechnet wird, R — den mittleren Radius der Erde, γ — den Mittelwert der Erdbeschleunigung.

Nehmen wir den Punkt P als Pol an, so ist

$$d\sigma = \sin\psi \, d\psi \, d\lambda.$$

Daraus ergibt sich:

$$N = \frac{R}{4\pi\gamma} \int_0^{2\pi} \int_0^{\pi} \Delta g S(\psi) \sin\psi \, d\lambda \, d\psi. \tag{2}$$

Dann finden wir nach partieller Integration der Formel (2):

$$N = \frac{R}{4\pi\gamma} \int_0^{2\pi} d\lambda \Big/ \Delta g Q(\psi) - \int \frac{\partial \Delta g}{\partial \psi} Q(\psi) \, d\psi \Big/_0^{\pi},$$

wobei

$$Q(\psi) = \int S(\psi) \sin\psi \, d\psi.$$

Da

$$Q(\pi) = Q(0) = -1,$$

wovon man sich leicht überzeugen kann, wenn man die Funktion $S(\psi)\sin\psi$ integriert, so folgt:

$$N = \frac{R}{4\pi\gamma} \int_0^{2\pi} \int_0^{\pi} \frac{\partial \Delta g}{\partial \psi} H(\psi) \, d\lambda \, d\psi, \tag{3}$$

wo

$$H(\psi) = -Q(\psi) - 1 = \sin^2\frac{\psi}{2}\left\{-\frac{4}{\sin\frac{\psi}{2}} + 5 + 6\sin\frac{\psi}{2} - 7\sin^2\frac{\psi}{2} + \right.$$

$$\left. + 6\cos^2\frac{\psi}{2}\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right\}. \qquad [(4)$$

Wenn wir in der Formel (3) $\frac{\partial \Delta g}{\partial s}$ durch $R\frac{\partial \Delta g}{\partial s}$ ersetzen, wo $\frac{\partial \Delta g}{\partial s}$ die Anomalie des Horizontalgradienten der Erdschwere in Richtung des maximalen Anstiegs von ψ genommen ist, erhalten wir schliesslich:

$$N = \frac{R^2}{4\pi\gamma}\int_0^{\pi}\int_0^{2\pi}\frac{\partial \Delta g}{\partial s}H(\psi)\,d\psi\,d\lambda. \qquad (5)$$

Den Wert von $\frac{\partial \Delta g}{\partial s}$ kann man mit Hilfe des Variometers von R. Eötvös ermitteln.

Mit Rücksicht auf den Effekt der örtlichen Anomalie des Horizontalgradienten auf die Erhöhung des Geoids, wollen wir anstatt der Veränderlichen ψ den Linearabstand r einführen; wenn wir denselben nach der Formel

$$r = 2R\sin\frac{\psi}{2}$$

bestimmen, erhalten wir:

$$H(r) = -\frac{2r}{R} + \frac{5}{4}\frac{r^2}{R^2} + \frac{3}{4}\frac{r^3}{R^3} - \frac{7}{16}\frac{r^4}{R^4} + \frac{3}{2}\left(1 - \frac{r^2}{4R^2}\right)\ln\left(\frac{r}{2R} + \frac{r^2}{4R^4}\right).$$

Bei sehr kleinen Werten von ψ kann man

$$\sin\frac{\psi}{2} = \frac{\psi}{2} = \frac{r}{2R}; \quad d\psi = \frac{dr}{R}$$

annehmen. Wenn wir $H(r)$ und $d\psi$ in die Formel (5) einführen, indem wir uns auf das kleine Gebiet in unmittelbarer Nähe des Punktes P beschränken, so erhalten wir:

$$N = \frac{1}{4\pi\gamma}\int_0^{2\pi}\int_0^{r_0} 2r\left\{-1 + \frac{5}{8}\frac{r}{R} + \frac{3}{8}\frac{r^2}{R^2} - \frac{7}{32}\frac{r^3}{R^3} + \right.$$

$$\left. + \frac{3}{4}\left(1 - \frac{r^2}{4R^2}\right)\ln\left(\frac{r}{2R} + \frac{r^2}{4R^2}\right)\right\} d\lambda\,dr.$$

Bei genügend kleinen Werten von r_0, kann man sich auf das erste Glied der obenerwähnten Formel beschränken, also

$$N = -\frac{1}{2\pi\gamma}\int_0^{2\pi} d\lambda \int_0^{r_0}\frac{\partial \Delta g}{ds}\,r\,dr.$$

Katheder für höhere Geodesie
Ingenieur-Bau-Institut.
Kharkow.

Eingegangen
d. 7. V. 1937.

ZITIERTE LITERATUR

[1] G. G. Stokes, On the Variation of Gravity at the Surface of the Earth.

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

BIOCHIMIE

DÉTERMINATION DE LA CONSTANTE DE DISSOCIATION DE LA THYRÉOGLOBULINE

Par I. A. SMORODINCEV (SMORODINZEW) et A. M. FELDT

(Présenté par A. A. Richter, de l'Académie, le 2. VI. 1937)

La constante de dissociation des ampholytes appartient au nombre des indices les plus importants, caractérisant la nature de la substance. Conformément à l'équation du degré de dissociation

$$\alpha = \frac{K}{K + [H^+]}$$

le degré de dissociation est égal à $^1/_2$ dans le cas où la valeur numérique de la concentration des ions d'hydrogène dans le milieu est équilibrée

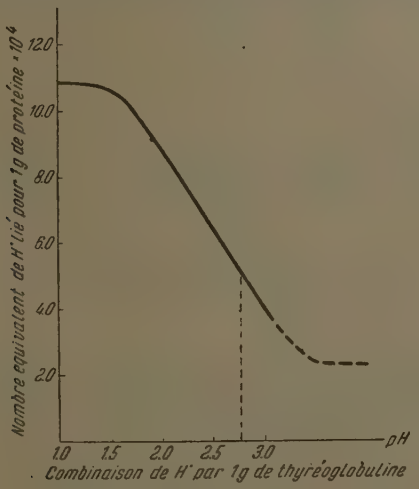


Fig. 1.

avec la valeur numérique de la constante de dissociation de la substance. Par conséquent, pour déterminer la constante de dissociation K, il suffit de déterminer la concentration des ions d'hydrogène au point où la substance est à moitié neutralisée (1).

Pour déterminer la neutralisation de la thyréoglobuline, nous avons adopté la méthode de liage de l'acide chlorhydrique par la protéine, vu que la courbe de liage de l'acide est analogue à la courbe de dissociation. Cette méthode a été employée pour déterminer les constantes de dissociation par Hitchcock (2), Pauli (3), etc.

L'étude a été faite avec la préparation de thyréoglobuline obtenue d'après le procédé de Heidelberger et Palmer (4), que nous considérons comme le meilleur (5). La préparation contient 0.00% de graisse et 0.05% de cendre.

On a préparé des mélanges de thyréoglobuline qui contenaient toujours la même quantité de protéine (10 g par litre) et des quantités variables d'acide chlorhydrique. La concentration des ions d'hydrogène était mesurée dans les mélanges au moyen de l'électrode piriforme à hydrogène de Michaélis et de l'électrode saturé de calomel. Le calcul des quantités d'acide chlorhydrique lié était effectué d'après la méthode de Cohn, également employée par Hitchcock (6).

Les données typiques d'une série de déterminations sont indiquées dans le tableau suivant:

Nos des expériences	1 Molarité de l'acide chlorhydrique m	2 Force électromotrice E, en volts	3 $\frac{E-0.2503}{0.0577}$, pH	4 Concentr. active d'H^+ dans le mélange a_H	5 Coeff. d'activité γ_H	6 Molarité de l'acide chlorhydrique libre $\frac{a_H}{\gamma_H}=m_H$	7 Molarité de l'acide chlorhydrique lié $m-m_H$	8 Nombre d'équival. d'H^+ liés par 1g de protéine, multiplié par 10^4, C_H
1	0.005	0.4210	2.96	0.00110	0.937	0.00117	0.00383	3.83
2	0.010	0.3939	2.49	0.00324	0.921	0.00351	0.00649	6.49
3	0.020	0.3651	1.99	0.01024	0.887	0.01154	0.00846	8.46
4	0.030	0.3512	1.75	0.01779	0.879	0.02023	0.00977	9.77
5	0.040	0.3420	1.59	0.02571	0.871	0.02951	0.01049	10.49
6	0.050	0.3351	1.47	0.03389	0.863	0.03927	0.01073	10.73
7	0.070	0.3247	1.29	0.05129	0.854	0.06005	0.00995	9.95
8	0.080	0.3212	1.23	0.05889	0.848	0.06944	0.01056	10.56
9	0.090	0.3178	1.17	0.06761	0.845	0.07977	0.01023	10.23

Dans la colonne 4 le chiffre a_H indique la concentration active de l'acide chlorhydrique, dans le mélange de protéine et d'acide chlorhydrique, correspondant au pH établi (antilogarithme). Dans la colonne 6 est calculée la molarité de l'acide chlorhydrique libre en tenant compte du coefficient d'activité: $m_H = \frac{a_H}{\gamma_H}$. Les coefficients d'activité γ_H sont empruntés au tableau de Scatchard (7). Cette méthode de calcul est basée sur l'admission de ce que le sel protéidique qui se forme exerce sur l'activité des ions d'hydrogène la même influence que l'acide correspondant aux ions liés d'hydrogène. Dans la colonne 8 est donné le nombre d'équivalents des ions d'hydrogène liés par 1 g de protéine, multiplié par 10^4.

Le titrage a montré qu'en cas de concentrations minimales de l'acide chlorhydrique, lorsqu' il est lié en plus grande partie par la protéine et que

les valeurs m_H sont insignifiantes, les titrages répétés donnent des résultats correspondant parfaitement. En augmentant la molarité de l'acide chlorhydrique, il en reste une partie relativement considérable non liée par la protéine et les écarts entre les titrages parallèles augmentent. Le tableau indique que jusqu'à 0.040 mol. gr. d'acide chlorhydrique par litre de thyréoglobuline à 1%, la quantité d'acide chlorhydrique lié (colonne 7) augmente régulièrement et progressivement jusqu'au maximum de 0.01049; ensuite elle semble se stabiliser, malgré la quantité d'acide chlorhydrique ajouté dépassant le double. La valeur moyenne d'une série de déterminations pour le nombre C_H: 10.49; 10.49; 10.73; 10.56; 10.23, est égale à 10.5, Le fait que la quantité d'acide chlorhydrique lié cesse d'augmenter tandis que sa concentration augmente dans le mélange, montre clairement que la saturation de la thyréoglobuline par l'acide chlorhydrique est déjà atteinte avec un pH de 1.59, quand la concentration d'acide chlorhydrique atteint 40 millimoles dans 1 litre de protéine à 1%. Par conséquent, la capacité d'un gramme de thyréoglobuline est égale à 10.5×10^{-4} d'équivalents du ion d'hydrogène, sa solution à 1% a une normale de 0.0105 et le poids de liage est égal à $\frac{10}{0.0105} = 950$. Nous avons obtenu pour le poids équivalent de cation de thyréoglobuline une valeur de même ordre que celles établies par plusieurs auteurs pour une série d'autres protéines [3].

Nous possédons maintenant toutes les données pour calculer la valeur approximative K, constante de dissociation de la thyréoglobuline: il faut trouver sur la courbe de liage de l'acide chlorhydrique par la thyréoglobuline la valeur de l'abscisse correspondant à l'ordonnée $5.25\left(=\frac{10.5}{2}\right)$. Elle est égale à 2.75; vu qu'en ce point, en vertu de l'équation du degré de dissociation, a lieu l'égalité $pK = pH$, la constante recherchée de dissociation de la thyréoglobuline est $K = 10^{-2.75} = 1.78 \cdot 10^{-3}$. Prenant en considération que la thyréoglobuline n'appartient pas aux ampholytes monobasiques, nous avons trouvé par ce procédé «la constante apparente de dissociation».

Comme nous le voyons, la constante de dissociation de la thyréoglobuline est assez considérable et, par conséquent, en cas de pH des tissus vivants la thyréoglobuline doit se trouver à l'état d'ionisation importante.

Conclusions

1. L'adhésion maximale des ions d'hydrogène ou la capacité de leur absorption par 1 g de thyréoglobuline est égale à $10.5 \cdot 10^{-4}$.

2. La normale de la solution de thyréoglobuline à 1% a été trouvée égale à 0.0105.

3. Le poids de liage de la thyréoglobuline ou son équivalent électrochimique est égal à 950.

4. La constante de dissociation de la thyréoglobuline $K = 1.78 \cdot 10^{-3}$.

Chaire de biochimie.
Institut chimico-technologique
de l'industrie de la viande.
Moscou.

Manuscrit reçu
le 2. VI. 1937.

LITTÉRATURE CITÉE

[1] И. А. Смородинцев, Успехи биолог. химии, **6**, 75 (1928). [2] D. J. Hitchcock, Journ. gen. Physiol., **4**, 733 (1922). [3] V. Pauli et E. Walko, Chimie colloïdale des substances protéidiques. [4] M. Heidelberger u. Palmer, J. biol. Chem., **101**, 433 (1933). [5] I. A. Smorodinzev et A. M. Feldt, **12**, 495 (1929). [7] G. Scatchard, J. Am. Chem. Soc., **47**, 696 (1925).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

GEOCHEMISTRY

THE «LAKE AGE» OF THE CASPIAN SEA AND ITS VOLUME AT THE TIME IT BECAME A CLOSED BASIN

By M. A. KLOČKO

(Communicated by N. S. Kurnakov, Member of the Academy, 29. IV. 1937)

There exist different views regarding the time when the Caspian Sea became a lake without an outflow. It is generally recognized that during the Permian period the Caspian Sea was in communication with the Ocean (1). On the other side, there are indications to the fact that in historic times, about 3000 years ago, the Caspian Sea was still connected with the Black Sea along the Manych valley (2).

In the present paper an attempt is made to solve the problems implied in its title, on the basis of a comparison of the saline composition of the Ocean, the Caspian Sea and the Volga, taking also into account the volume of the Caspian Sea and the quantity of salts which are yearly brought down to it by rivers.

In the table below are given data on the two most easily and accurately determinable constituents of the saline mass of the water—chlorine and SO_4 group, as well as the relation SO_4/Cl and also the total percentage of the content of salts in the water of the three above mentioned reservoirs.

	The World Ocean (3)	The Caspian Sea(4)	The Volga (3)
SO_4	0.269%	0.298%	52 mg/l
Cl	1.935%	0.524%	8.3 »
Relation SO_4/Cl	0.139	0.569	6.26
Tolal percentage of the dissolved salts	3.50 %	1.260%	—

In comparing the water of the Caspian Sea with that of the Ocean one's notice is attracted by the lower salinity, and the relatively higher content of SO_4 in the former. The lower concentration of the Caspian water may be accounted for by the fact that at the time of separation from the Ocean the volume of the water mass of the Caspian Sea was smaller than it is now, and thereafter its water, oceanic by its composition, was diluted with river-

water. This was also the reason of an increase of the relative content of sulphides in the water of the Caspian Sea, as in river-water sulphides usually predominate over chlorides.

The main affluent of the Caspian Sea is the Volga which provides $^3/_4$ of water that rivers supply to this sea. Considering that the composition of the other rivers does not greatly differ from that of the Volga and using the data given in the above table, we can approach the solution of our problem in the following manner

The water of the Volga at Saratov contains, according to the analyses made by V. P. Radistchev, 52 mg SO_4 and 8.3 mg Cl per liter (an average for 1924) ([5]). Assuming the Volga is supplying the Caspian with 290 km^3 of water yearly and the other rivers of its basin with 90 km^3 ([4]), and the composition of the latter approaches that of the Volga, we shall arrive at the annual river influx into the Caspian Sea as being equal to 380 km^3 of water, which increases the content of chlorine in this sea by $380 \cdot 8.3 \cdot 10^3$ t yearly as $1 \frac{mg}{l} = 10^3 \frac{t}{km^3}$.

In the same way we shall get the annual increase in SO_4 which is equal to $380 \cdot 52 \cdot 10^3$ t.

1 km^3 of water of the Ocean contains $0.269 \cdot 10^7$ t SO_4 and $1.935 \cdot 10^7$ t Cl, as 1% gives 10^7 t per 1 km^3 (if the specific gravity is assumed to be equal to 1).

Let us assume that the volume of the Caspian Sea at the time of its separation from the Ocean was equal to X km^3 and that this event occurred Y years ago. The volume of the Caspian Sea is at present equal, according to data of N. M. Knipovich, to 79 300 km^3 ([4]).

Comparing the absolute quantities of SO_4 and Cl contained in the Caspian Sea to-day with those at the time when this sea became a closed basin (it being assumed that the composition and concentration of salts in the Caspian Sea and in the Ocean were the same at the moment of separation) and attributing the difference in the quantities of chlorine and SO_4 at these two moments to the supply of salts by the rivers whose composition is analogous to that of the Volga, we can form two equations with two unknowns.

Indeed: at the moment of separation, the Caspian Sea contained $X \cdot 1.935 \cdot 10^7$ t chlorine, and now it contains $0.524 \cdot 10^7 \cdot 79\,300$ t chlorine—the difference between these two quantities has been supplied by rivers, bringing down chlorine during Y years at $380 \cdot 8.3 \cdot 10^3$ t yearly, i. e. $0.524 \cdot 10^7 \cdot 79\,300 - X \cdot 1.935 \cdot 10^7 = Y \cdot 380 \cdot 8.3 \cdot 10^3$. This is an equation on the quantity of chlorine.

Just in the same way we shall form the following equation on the quantity of SO_4:

$$0.298 \cdot 10^7 \cdot 79\,300 - X \cdot 0.269 \cdot 10^7 = Y \cdot 380 \cdot 52 \cdot 10^3. \qquad (2)$$

After some changes, and division of each equation by 10^6, we shall get:

$$19.35\,X + 3.154\,Y = 415\,700, \qquad (1')$$

$$2.68\,X + 19.76\,Y = 236\,300. \qquad (2')$$

After solving these equations, we shall get:

$$X = 19\,985; \quad Y = 9\,234,$$

i. e. the Caspian Sea lost its connection with the Ocean about 9000 years ago, its volume having been at that time $^1/_4$ of its present volume.

It is understood that the above calculations should be interpreted as a matter of principle, and the obtained data should be regarded as the first rough approximation. The fundamental argument against the above computation is that we know neither the quantity of the river waters which were yearly flowing into the Caspian Sea, nor their composition during the past thousands of years. There was a time when to the Caspian basin belonged (although partially) such a river, abounding in water and rich in salts, as Amu-Darya which for the last five centuries has been wholly flowing into the Aral Sea ([6]).

The above objection as well as the others stated below do not weaken the importance of the method applied in this paper, as a general principle. As to the quantity of water in the river supply, if this exceeded the present day river supply of the Caspian Sea, it means that a smaller space of time would have been required (if the saline composition is assumed to be constant) in order to bring the Caspian Sea into its present state. As regards the composition, i. e. essentially the content of SO_4 and Cl, it may be considered that in the past thousands of years rivers were richer in salts, as at the present time we are nearer to the end of the process of washing out salts from the soil than we were at that time. This also leads to a reduction of the number of years elapsed since the separation of the Caspian Sea. Furthermore, we are not taking account of the quantities of salts which were left by the Caspian Sea in the soil when receding by its «regression», but if we should manage to determine these we should have to add them to the quantity of salts contained at the time of separation which, again, would decrease the difference between the quantities of salts contained in the Caspian Sea at present and those contained in it at the time of separation. These considerations lead us to the conclusion that the figure—9234 years—which has been computed by us, is the superior limit for the period of time in the course of which the Caspian Sea has existed as a lake. There is no doubt that if computations are carried out on the basis of more accurate data, this should lead to a reduction of the above figure and, at the same time, to an increase of the value found for the volume of the aqueous mass at the time of separation of the Caspian Sea from the Ocean. The probability of the fact that the Caspian Sea before separation may have already contained a lesser percentage of salts in comparison with that of the Ocean is another argument working for the reduction of its «lake age».

Institute of General and Inorganic Chemistry.
Academy of Sciences of the USSR.
Moscow.

Received
29. IV. 1937.

REFERENCES

[1] А. П. Карпинский, Очерки геологического прошлого Европейской России (1919). [2] С. А. Ковалевский, Лик Каспия, ч. 1, Тр. Геолого-развед. конт. Азнефти, вып. 2 (1933). [3] F. W. Clarke, U. S. Geolog. Surv. Bull., 770, 1—760 (1924). [4] Н. М. Книпович, Тр. Каспийский экспедиции 1914—1915 гг.; Гидрологическое исследование в Каспийском море в 1914—1915 гг. (1921). [5] В. П. Радищев, Работы Волжск. биол. станц., VIII, № 4—5, 231—266 (1926). [6] Л. Берг, Аральское море (1908).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

SOIL SCIENCE

ON THE PROBLEM OF THE GENESIS OF VIVIANITE

By K. S. ANDRIANOV and A. I. SMIRNOV

(Communicated by V. I. Vernadsky, Member of the Academy, 10. V. 1937)

Iron phosphates have of late come in for a good deal of attention as one of the varieties of phosphate fertilizers.

The fairly high efficiency of vivianite has been shown in a number of papers ([5,7,8,14,17,20]), while prospecting made by the Scientific Institute for Fertilizers and Insectofungicides (NIUIF) in White Russia has proved its frequent occurrence in lowland bogs*.

Vivianite, a ferrous phosphate $Fe_3(PO_4)_2 8H_2O$, and a number of its derivatives (kerchenites, beraunite, picite, etc.) are mainly encountered in peat bogs of the lowland type ([6, 12, 22]). In addition to this, vivianite has been found in iron ore deposits ([10]), in alluvial sediments ([9]), in podzol, swampy and flood-plain soils ([2,4]), and is also mentioned in the list of minerals occurring in pegmatite veins ([16]) deposited into the hydrothermal phase. Three groups of hypotheses are current with regard to the origin of peat bog vivianites.

A. Gaertner ([24]), G. Potonier ([11]), B. P. Krotov ([9]), G. Puchner ([12]) and others consider vivianite to be the result of the interaction of the phosphoric acid liberated in the decomposition of organic substances with the circumambient iron compounds. The deposition of vivianite involved in the process occurs either in the immediate vicinity of such organic remains, or at some distance therefrom, forming lenticular inclusions or streaks in the body of the peat. Unlike these writers, G. Schreiberg ([19]), and later A. N. Orlov connect the formation of vivianite with the phosphorus carried by ground waters. According to G. Schreiberg, vivianite is formed as a result of a reaction between the calcium phosphate leached out of the rock by the action of the ground waters, and the siderite which occurs in bogs. According to A. N. Orlov, the deposition of vivianite is due to an increased concentration of the solutions consequent upon the drying-up of bogs.

A third hypothesis is due to W. R. Williams ([2]), who seeks the origin of phosphorus in soil processes. In the formation of podzol, the phosphates are leached out of the soil and collected in the Ortstein horizon. The destruction of the latter gives rise to calcium phosphate, which finds its way into marshy grounds adjoining terrace formations, where a reaction of exchange with the sulphates of iron gives rise to vivianite.

In our opinion, all of these theories are vitiated by a common flaw, since they all emphasize the problem of the sources of phosphorus, while leaving

* During the season of 1936, the vivianite party of Scientific Institute for Fertilizers and Insectofungicides (NIUIF) (headed by A. N. Orlov) brought to ligth about 1 000 000 m^3 of vivianitized peat, with an average P_2O_5 content of 5 to 15%.

practically aside those physico-chemical conditions that have determined the deposition of the iron phosphates and suggesting no adequate explanation for the mechanism of accumulation of phosphorus in bogs, in sometimes fairly large quantities. In our opinion, however, it is the peculiarities of the solubility of iron phosphates that play a decisive part in the accumulation of vivianite.

Teakle ([25]) has shown the solubility of ferric phosphate to vary sharply with the reaction of the medium. The phosphate shows a minimum of solubility at pH=3.0, any increase or decrease of acidity causing a rapid growth of solubility. Similar results were also obtained by Doughty ([23]). In his experiments, with a ratio of Fe: PO_4=1 : 1, the lowest solubility of the iron phosphate was observable at pH=4.0, when practically the whole of the PO_4 was found to be in the precipitate. On increasing the acidity to pH=2.0, the whole of the phosphate passed into solution. When the solution was neutralized, the hydrolysis of $FePO_4$ set in and proceeded very rapidly at pH=6.0. When, in the above experiments, the Fe : PO_4 ratio was increased to 10 : 1, the hydrolysis of the iron phosphate was inhibited until pH reached the value of 10.0.

The solubility of ferrous phosphates was studied by S. R. Carter and N. H. Hartschorn ([21]) who investigated the $FeO—P_2O_5—H_2O$ system. It is a matter for regret that the system was studied only at high concentrations of P_2O_5 (7.38—57.51%), acid ferrophosphates being obtained. A. P. Višniakov ([3]) also studied the solubility of natural vivianite in centinormal solutions of NaOH+HCl. In these experiments, the minimum solubility of the P_2O_5 of the vivianite was found to lie at pH=4.5. On either side of this point, the solubility of P_2O_5 showed a sharp increase.

The papers just mentioned therefore clearly indicate that the minimum solubility of ferrous phosphates lies in the acid interval (2.5—4.5). In neutral or alkaline media, iron phosphates become hydrolized and decompose, the hydrates being separated, and the phosphoric acid passing into solution. Only a considerable excess of iron causes the hydrolysis to be retarded until higher values of pH are reached.

These experiments, however, do no more than afford a very rough-and-ready representation of the conditions of solubility, a number of highly important questions remaining unsolved (minimum concentrations of Fe and PO_4 necessary for the deposition of phosphates to begin, the influence of the excess of iron on the solubility of vivianite, etc.). Some conclusions however, may even now be drawn as to the conditions of the occurrence of vivianite in nature.

First and foremost it should be recognized that the accumulation of iron phosphates is mainly possible in such portions of the earth's crust that have a stable acid reaction. Wherever the reaction is neutral or alkaline, no iron phosphates can occur unless the excess of iron is very high.

The most favourable combination of these conditions (an acid medium joined to a fairly high iron content) with a simultaneous supply of phosphates would appear to be afforded by ground-fed bogs, so that, considered in their relation to the general circumrotation of phosphorus, bogs may be said to act as concentrators trapping iron phosphates.

It is well known ([6]) that the most acid reaction is possessed by upland bogs, where pH sometimes falls below 3.0. As to ground-fed lowland bogs, the reaction is usually a less acid one, becoming even alkaline in the case of an accumulation of lime. It is not, however, any difference in the processes

of peat formation that appears to be responsible for the lower acidity of lowland peat bogs, but the secondary processes of neutralization of the acid products of peat formation due to the carbonates brought by the ground waters.

A necessary concomitant of peat formation is a considerable accumulation of so-called humic acids, the percentage of which, according to Puchner[12] may reach 50% of the weight of the peat. A. Schmuck[18], Stadnikov[15] and others are of the opinion that several carboxyl groups are present in humic acid, «the acidity of humic acid being thus a true acidity». Still, according to V. I. Vernadsky[1], the acidity of paludal waters is too high for humic acids, and is therefore probably due to some stronger organic acids. This point of view is endorsed by Rediger[13], who establishes the dependence of pH on the so-called volatile residue (water soluble organic substances).

The water-soluble and the humic acids are partly neutralized by the bases liberated in the mineralization of organic remains. The main bulk, however, of the mineral components of peat is brought into the bog by ground and surface waters. The latter do not play an important part in the mineral supply of bogs, owing to the low concentration of the substances dissolved in them, the short period of life of surface streams and the prevalence of an ascending flow of water in bogs. Underground waters should be of paramount importance in the mineral supply of lowland bogs.

According to V. I. Vernadsky's [1] scheme, over 30 elements have been quantitatively identified in underground fresh waters and in wells, spring and sources, the most interesting among them being the ever present phosphorus, calcium and iron.

According to V. I. Vernadsky, most of the phosphorus is present as PO'''_4 or HPO''_4 ions. In terms of P_2O_5, the percentage of phosphorus in underground waters is as high as 1. 4 mg/l. Iron is transmitted as a ferrous bicarbonate, organic mineral complexes or colloidal solutions of the hydroxide. Expressed in terms of Fe_2O_3, the iron content may be as high as 30 mg/l. Underground waters contain calcium as bicarbonates at the rate of up to 200 mg CaO per litre.

On penetrating into a peat bog, the underground waters carrying matter in solution must alter both their own composition and that of the peat.

The humic and other acids must in the first place destroy the carbonates, liberating CO_2, while calcium and iron form humates which partly accumulate in the bog and are partly carried away by the bog waters. Phosphorus and iron, if present in sufficient concentrations, may be precipitated as vivianite due to the acid reaction.

Subsequently, under the action of a continued influx of ground waters, the acid water-soluble compounds are carried away, the humic acids being converted into humates and the acidity of the medium being reduced. The solubility of iron phosphates increases with the decrease of acidity, so that the vivianite deposited in the first period of life of a lowland peat bog must at a certain value of pH, re-dissolve and migrate to a part of the peat bog having a more acid reaction.

Finally, when the peat has become sufficiently neutralized, conditions may arise that will not hinder the precipitation of calcium and iron carbonates, if the concentration of the latter is sufficient.

Thus, if a peat bog is invaded by ground waters rich in phosphates and carbonates of calcium and iron, four distinct zones differing as to the reaction of the medium and their mineral content, must be formed, beginning from the spring head towards the water reservoir.

In the most neutralized portions, near the springs, the carbonate zone will lie, where lime may be deposited if the water is rich in calcium carbonates, or siderite if the water is chalybeate.

In the more acid zones, amid the carbonates, mainly amid those of iron, there will be encountered separate grains of vivianite, which had formerly occurred here as a layer and has not yet been dissolved.

Farther away from the spring, in a zone characterized by a stable acid reaction, where the conditions are most favourable to the accumulation of iron phosphates, the peat will contain inclusions and streaks of vivianite.

Still farther away lies peat which bears no visible traces of mineralization.

In course of time, as the peat bog becomes more and more neutralized by the influx of ground waters, the four zones move forward in the direction of the flow of water. Vivianite is dissolved and re-deposited wherever the reaction is still acid enough. Ultimately, the vivianite zone must be shifted to the ground water level in the bog, or to the level of the water reservoir draining the bog, whereafter any lowering of the water level gives rise to the oxidation of vivianite.

Such is, in our opinion, the scheme of the formation of vivianite in lowland peat bogs, which has so far been fairly well confirmed by the example of deposits examined in White Russia in 1936.

In the deposits so examined, vivianite is encountered only in lowland bogs fed by ground waters, and occurs in the shape of lenses or streaks in the body of the peat. The lenses of vivianitized peat have the shape of layer-like intrusions associated with those layers of peat that are the least impermeable to water. The lenses of vivianitized peat occur as a rule in the vicinity of depressions in the bottom of the peat bog where springs well up.

In the deeper portions of the peat bog, in the vicinity of spring heads, accumulations of lime or siderite are commonly encountered, above which in the vertical section, or below in the direction of flow of the bog waters, vivianite may occur. In bogs fed by hard carbonaceous waters or by those rich in lime or siderite only stipplings of vivianite are encountered.

Research Institute for Fertilizers
and Insectofungicides.

Received
7. IV. 1937.

REFERENCES

[1] В. И. Вернадский, История минералов земной коры, т. II. История природных вод, ч. I, вып. II (1934). [2] В. Р. Вильямс, Почвоведение, ч. I, II (1926). [3] А. П. Вишняков, Труды Ленинградского отдела ВИУА, вып. 36 (1935). [4] К. Глинка, Почвоведение (1933). [5] Л. Горяинова, Тр. Вс. ин-та торфа 62,65—74 (1932). [6] В. С. Доктуровский, Торфяные болота (1935). [7] Ф. Н. Доминиковский, Химизация соц. зем., № 7, 68—75 (1934). [8] Ф. П. Доминиковский, Бюлл. ВАСХНИЛ, № 10 (1936). [9] Б. П. Кротов, Записки Минер. о-ва, сер. II, 53, вып. 2 (1924). [10] С. П. Попов, Тр. Геол. муз Акад. Наук, IV, вып. 7 (1910). [11] Г. Потонье, Сапропелиты. Серия «Нефтяное и сланцевое хозяйство» (1920). [12] Генрих Пухнер, Торф (1929). [13] В. Р. Редигер, Научн.-агроном. журнал, № 5—6 (1929). [14] Н. Д. Смирнов, Минерал. удобрения и инсекто-фунгисиды, № 3 (1935). [15] Г. Л. Стадников, Химия торфа (1930). [16] А. Е. Ферсман, Пегматиты (1932). [17] Э. Шиперко, Итоги работы Минской болотной опытной станции за 1918—1928, г. Минск (1930). [18] А. Шмук, Бюлл. почвоведа, № 5—7 (1930). [19] Г. Шрейберг, Вестник торф. дела, № 1, 56—73; № 4,72—88 (1918). [20] И. С. Шулов. Из результатов вегетац. опытов и лабор. работ за 1908—1909 г., VI, 190 (1911). [21] S. R. Carter and N. H. Hartschorn, Journ. Chem. Soc., 371, (1926). [22] Doelter, Handbuch der Mineralchemie, 402—412 (1918). [23] J. L. Doughty, Soil Sci. XXIC, No 1, 23 (1930). [24] Gaertner, Arch. des Ver. d. Freunde d. Naturges. in Mecklenburg, 51 (1898). [25] L. J. H. Teakle, Soil Sci., No 2 (1928).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

GENETICS

FREQNENCY OF LETHALS IN THE *X*-CHROMOSOME DUE TO THE IRRADIATION OF SPERMATOZOA IN THEMALES AND IN THE SPERMATHECAE OF THE FEMALES OF *DROSOPHILA MELANOGASTER*

By P. O. SITKO

(Communicated by A. A. Sapehin, Member of the Ukrainian Academy, 8. V. 1937)

The question discussed in this paper is that of the influence of sex and of irradiated plasm on the mutation process in spermatozoa. The object of the work was to verify the conclusions arrived at by Giulbekian.

M e t h o d. The experiments were performed on *Drosophila melanogaster*. Both for the experiment (spermatozoa X-rayed in the spremathecae of the females) and the control (spermatozoa X-rayed in the males), normal males and virgin *ClB* females were prepared simultaneously. 24 hours before exposure, half of the females were crossed with the prepared males. In order to ensure equality of X-ray dosage for the spermatozoa in the spermathecae of the females and those in the body of the males, both the males and the fertilized females were placed together in gelatine capsules for irradiation. After irradiation, the fertilized females were separated from the males and placed on the medium, while the control males were crossed with the second half of the virgin females. In order to eliminate the influence of the ageing of the irradiated sperm, only the offspring hatched from eggs laid during the first 24 hours were investigated both in the experiment and in the control. In the control, the males were thrown away in 24 hours, only the fertilized females being kept. In addition to this experiment, offspring obtained from the sperm 10 days after X-raying was also investigated. The dosage was 3000 r. The method used in detecting lethals was the well-known *ClB* method. No account was taken of cultures having fewer than 20 females.

Results of Experiments

Table 1

Place of irradiation of spermatozoa	Experiment I			Experiment II			Both together		
	Number of chromosomes	Number of lethals	Percentage of+lethals	Number of chromosomes	Number of lethals	Percentage of+lethals	Number of chromosomes	Number of lethals	Percentage of+lethals
In the tests of the males	470	28	5.95 ± 1.09	415	23	5.54 ± 1.12	885	51	5.76 ± 0.78
In the spermathecae of the females	493	27	5.47 ± 1.02	456	28	6.14 ± 1.12	949	55	5.79 ± 0.76

As may be seen from Table 1, there is no difference in the freqúency of lethals between the spermatozoa X-rayed in the spermathecae of the females, and those X-rayed in the bodies of the males. The frequency of lethals in

experiment I and experiment II taken together is practically the same, being 5.79+0.76 in the experiment, and 5.76+0.78 in the control. Table 2 shows the results of the experiments on the 10th day after the X-raying of the sperm.

Table 2

Place of irradiation of spermatozoa	Number of chromosomes analysed	Number of lethals	Percentage of lethals
In the tests of the males .	541	42	7.76 ± 1.15
In the spermathecae of the females	342	28	8.18 ± 1.76

As may be seen from this table, neither in this case has any difference arisen in the frequency of lethals between the spermatozoa X-rayed in the spermathecae of the females, and those irradiated in the bodies of the males. At the same time, an interesting fact came to light, viz., that both in the experiment and in the control the frequency of lethals is higher on the 10th day after irradiation than it is on the day following irradiation; in the experiment we have 8.18±1.76 on the 10th day after irradiation, and 5.79±0.76 on the day following irradiation; in the control, the figures are 7.76±1.15 and 5.76±0.78 respectively. In continuing the experiments I succeeded in obtaining in the control, 105 cultures on the 20th day after irradiation, the number of lethals among them being 11, or 10.47%±2.99. It is evident therefore that the increase of mutability in ageing irradiated spermatozoa is not purely fortuitous. The above experiments show that there is no difference in mutability between the spermatozoa irradiated in the spermathecae of females and those irradiated in the body of males, either on the day following irradiation, or ten days after. There is, however, a difference in the mutability of spermatozoa on the 1st, 10th and 20th day after irradiation, both in the experiment and in the control.

Discussion. The above experiments, unlike those of Giulbekian, have failed to disclose any influence of the biological peculiarities of either males or females, or of the irradiated plasm, on the frequency of lethals in the spermatozoa. It is to be supposed that Giulbekian's results are to be accounted for by his methods of work rather than by any biological peculiarities of either the males or the females, or of the irradiated plasm. In his experimental series, the females had received spermatozoa before irradiation, so that the offspring obtained a few days after irradiation may have been affected by the increase of mutability due to the ageing of spermatozoa as disclosed by our experiments; as to the control, the females were all the time kept together with the males and received new doses of sperm, which would account for non-increased mutability.

Conclusions. The frequency of lethals in the X-chromosome of irradiated spermatozoa is the same whether the latter are irradiated in the spermathecae of the female or in the body of the male.

The frequency of mutations increases in ageing irradiated spermatozoa. In the experiment, the percentage of lethals is 5.79±0.76 on the day following irradiation, and 8.18±1.76 ten days after. In the control, the percentages are 5.76±0.78 and 7.76±1.15 respectively, and they are 10.47±±2.99 20 days after irradiation.

Section of Genetics. Institute of Zoology and Biology. Academy of Sciences of the Ukrainian SSR.

Received 8. V. 1937.

REFERENCES

[1] Биологический журнал, V, вып. 1 (1936).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

PLANT PHYSIOLOGY

THE BEHAVIOUR OF NITROGENOUS SUBSTANCES DURING THE YAROVIZATION OF PLANTS

By I. N. KONOVALOV and I. E. ROGALEV

(Communicated by A. A. Richter, Member of the Academy, 8. V. 1937)

One of the most important problems confronting plant physiology is that of finding the physiological explanation of the qualitative changes, occurring in plants during yarovization, owing to which winter plants acquire the property of «earing» the same year they are sown. On the elucidation of this problem depends the success of theoretical and practical work on controlling the development of plants. This problem is therefore of great agricultural importance.

A considerable amount of work has been done to elucidate the nature of the qualitative changes occurring in plants during yarovization, but, nevertheless, the problem has not been solved. Some workers [as, for instance, P. I. Demkovskij (3), N. V. Savostin and M. M. Okuncov (10), and others] attribute these changes to the activity of enzymes. Others believe the development of plants to depend on hormones [T. A. Krasnoselskaja-Maximova(6), N. G. Cholodnyj (12), M. Ch. Čajlachjan (13,14) and B. S. Moškov (7)], and, lastly, still others associate the qualitative changes occurring during yarovization with changes in the protein complex of the plant organism [A. A. Richter (8,9), V. A. Rancan and M. V. Pekker (9), A. M. Bassarskaja (1,2) and I. A. Filippenko (11)].

According to the conclusions of Academician A. A. Richter and his collaborators (8,9), the changes in the activity of enzymes, though presenting certain interest, do not characterize the qualitative changes occurring in plants during yarovization. The role of hormones is also but very little known, and there are no data as to their being the decisive factor in regard to the qualitative changes occurring in plants during yarovization.

The results of investigations associating the qualitative changes occurring in plants during yarovization with protein substances lead to quite different conclusions. The investigations of A. A. Richter and his collaborators (8,9), A. M. Bassarskaja (1,2), and I. A. Filippenko (11) have shown that the yarovization of plants is attended with changes in the physicochemical properties of the protein complex of the plasma. These investigations appear to us to be of especial interest, as we believe that they give an idea of the qualitative changes independently of what underlies them,— the action of hormones or other factors controlling the normal development of plants.

This paper deals with one of the investigations undertaken on A. A. Richter's initiative for the purpose of studying the behaviour of nitrogenous substances in plants during yarovization. In the system of a series of methods tried by us for investigating the qualitative changes associated with yarovization, the study of individual groups of nitrogenous substances might give us an idea of the part played by the protein complex of plants during yarovization and point out the way for further investigations in this respect.

This work was carried out in 1936. The plants experimented on were the Ukrainka winter wheat and the Hordeiforme 432 spring wheat. Both the winter and spring varieties chosen for our experiments are forms strongly responding to yarovization. The winter wheat was yarovized for 50 days and the spring variety for 18 days. The seeds were yarovized in a constant temperature thermostat after the method elaborated by Academician Lysenko. Up to the «tillering» stage, the green plants were grown under greenhouse conditions and were then transferred into a cold chamber, where they were kept at a temperature of +2, +5°C under continuous electric illumination.

For the purpose of elucidating the dynamics of the changes occurring in nitrogenous substances during the yarovization of plants, which proceeds in wheat, as is well known, under a complex of factors with low temperature as the leading factor, we also studied, as a control, the dynamics of the changes occurring in nitrogenous substances under optimum conditions of seed germination. With this object in view, we cultivated seeds in a dark thermostat on moist filter-paper at temperature 25—26°C and took samples at different stages of germination for the determination of nitrogenous substances.

Samples for analysis of nitrogenous substances were collected on the 11th, 22nd, 33d, 44th and 50th day in yarovizing winter wheat and on the 6th, 12th and 18th day in yarovizing spring wheat. Analyses of nitrogenous substances in the specimens grown at high temperatures were made on the 5th, 11th and 14th day. The quantity of nitrogenous substances in the dry seeds and in the soaked seeds before yarovization was also determined.

The following forms of nitrogenous substances were estimated: total nitrogen; nitrogen of water-insoluble proteins; nitrogen of water-soluble proteins; nitrogen of the amide group; nitrogen of the amine group; nitrogen of the free ammonia. Determination of the total nitrogen, the water-insoluble protein nitrogen and water-soluble protein nitrogen was made after the Kjeldahl macromethod. The amine nitrogen was determined after the van Slyke method, the amide nitrogen after the Sachs-Wickerey method, and the ammonia nitrogen in vacuum. A description of the above methods may be found in the handbooks by Prof. N. N. Ivanov ([5]) and Prof. A. R. Kizel ([7]).

The development of both the Ukrainka winter wheat and the Hordeiforme 432 spring wheat was accelerated by the yarovization. The spring wheat shortened the duration of its earing and ripening stages by 4 days. The yarovized winter wheat reached maturity in 110—115 days, while the non-yarovized did not reach the «earing» stage until autumn. This shows that the operation of yarovizing the wheat came off successfully in our experiments and that, consequently, our findings are well grounded.

As regards the changes experienced by the nitrogenous substances of the plants during yarovization, we obtained the following data. It was found that, although the total amount of nitrogenous substances was more or less constant during the process of yarovization, these substances underwent considerable inner changes.

The amount of total nitrogen does not vary much during the process of yarovization. A certain tendency towards an increase of its content observed in almost all the experiments appears to be due to a decrease in the carbohydrate reserve, which forms the source of energy during the process of yarovization. This rise in nitrogen content is most probably only relative. The amount of the insoluble protein nitrogen decreases considerably, both during the yarovization of seeds and during the yarovization of green plants. The increase in the amount of the soluble protein nitrogen during the process of yarovization corresponds to a decrease in the amount of the insoluble protein nitrogen. The amount of non-protein forms of nitrogen also changes during yarovization.

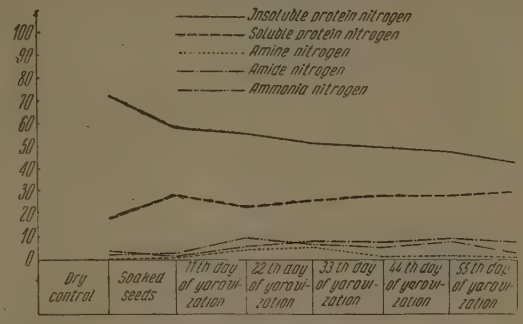


Fig. 1. Changes of Nitrogenous Substances Occurring during Yarovization of the Seeds of Ukrainka Winter Wheat

For the sake of greater clarity, two diagrams are shown below.

The data on the changes of nitrogenous substances occurring during the yarovization of the seeds of spring wheat show the same picture and are therefore not given here. It should be noted that in our method of determining the amount of nitrogenous substances not all the nitrogen of the amine group is taken into account and that, consequently, fractional determination of nitrogen content gives figures lower than the total nitrogen.

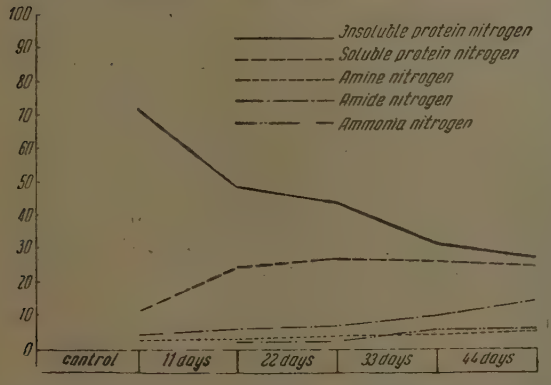


Fig. 2. Changes of Nitrogenous Substances Occurring during Yarovization of the Green Plants of Ukrainka Winter Wheat

A comparison of the data on the changes of nitrogenous substances occurring during yarovization with those during the germination of seeds at high temperatures, when no yarovization can occur, gives interesting results. The data on the changes of nitrogenous substances in seeds geminating at high temperatures are shown in Fig. 3.

A comparison of the data presented in the diagrams shows that there exists a great difference between the changes of nitrogenous substances of wheat seeds occurring during yarovization and those occurring during germination at high temperature. During seed germination at high temperature, a considerable part of the protein decomposes into the end products

of decomposition, while this is not the case in yarovization. Thus, on the 5th day of germination of the seeds of Ukrainka winter wheat, at high temperature, the non-protein forms of nitrogen (the aggregate nitrogen of the free ammonia and of the amine and amide groups) constitute 20.38% of the total nitrogen, while on the 11th day this figure reaches 36.51%. During the yarovization of seeds, decomposition never attains such a degree, notwithstanding the long duration of this process; on the contrary, the content of non-protein forms of nitrogen, on reaching a certain maximum, after the soaking of the seeds prior to yarovization, begins to decrease and does not then exceed 10% of the total nitrogen. Together with the decrease in the amount of the non-protein forms of nitrogen, there occurs a certain increase in the nitrogen of the water-soluble protein. It appears that during yarovization there occurs re-synthesis of nitrogenous substances, and this fact constitutes the principle difference between the changes of nitrogenous substances occurring during yarovization and those occurring during seed germination under optimum conditions.

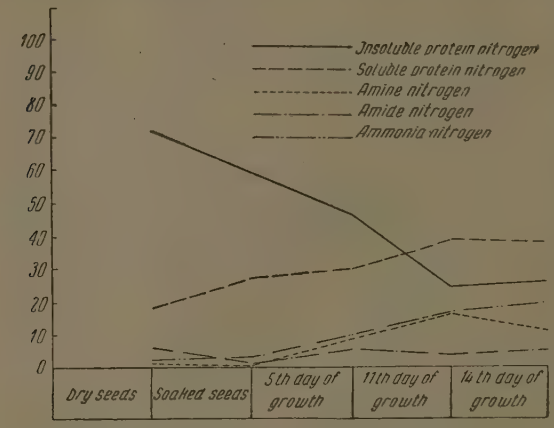


Fig. 3. Changes of Nitrogenous Substances Occurring During Germination of the Seeds of Ukrainka Winter Wheat at 25—26° C

It is therefore possible to affirm that the qualitative changes occurring in plants during yarovization are very closely associated with changes of the protein substances. The preservation of the nitrogenous substances in their major part in the form of proteins ensures the possibility of a correlation bewteen the plant's internal work of plasma differentiation and of yarovization and thus guarantees the full development of the plant during the initial stages of its development.

Received
8. V. 1937.

REFERENCES

[1] А. М. Бассарская, Яровизация, № 6 (9) (1936). [2] А. М. Бассарская, Семеноводство, № 3 (1934). [3] П. И. Демковский, Бюллетень яровизации № 2, 3 (1932). [4] Н. Н. Иванов, Методы физиологии и биохомии растений (1932). [5] А. Р. Кизедь, Практическое руководство по биохимии растений (1934). [6] Красносельская-Максимова, Труды по прикладной ботанике, XXVII, вып. 5 (1931). [7] В. С. Мошков, Соц. растениеводство, № 19 (1936). [8] А. А. Рихтер, Природа, № 2 (1934). [9] A. A. Richter, V. A. Rancan a. M. V. Pekker, C. R. Acad. Sci. URSS, № 2 (1933). [10] М. М. Окунцов, Н. В. Савостин, Труды Томского государственного университета, 86 (1934). [11] I. A. Filippenko C. R. Acad. Sci. URSS. III, № 4 (1936). [12] Н. Г. Холодный, Природа № 8, 9 (1933). [13] M. Ch. čajlachjan, C. R. Acad. Sci. URSS, IV, № 2 (1936). [14] M. Ch. čajlachjan, C. R. Acad. Sci. URSS, III, № 9 (1936).

Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS
1937. Volume XVI, № 1

VIROLOGY

A NEW DISEASE OF THE ONION (*ALLIUM CEPA*)

By V. L. RYŽKOV and A. M. VOVK

(Communicated by G. A. Nadson, Member of the Academy, 27. IV. 1937)

In the end of May 1936, we discovered a mosaic disease of the onion, occurring in a great many plants in the Agricultural Experimental Station in Alexeyevka, near Kharkov. The disease causes a mosaic pattern on the leaves and attacks the inflorescence.

The mosaic pattern on the leaves was of various kind. In some instances, we saw minute, more or less elongated specks; in others, more or less broad light-green or cream-coloured bands were to be observed (Fig. 1). When called forth by artificial infection, the disease starts with solitary yellow spots of an irregular shape, the bands appearing later on. Under the influence of the disease, the inflorescences become of a loose texture for the peduncles elongate in an irregular way. The perianth of the diseased flowers is gamopetalous, it elongates and has long twisted lobes. The androecium is depressed in its development and sometimes transforms into leaflets. The pistils not infrequently separate into more or less long leaflets. Sometimes, instead of flowers, small bulbs develop on long peduncles. The diseased plants either do not bear seeds at all or else the latter are few. While the disease progresses, the mosaic may be masked. Bulbs collected from outwardly sound plants often, on germinating, yield mosaic leaves.

Fig. 1. An onion affected with mosaic disease.

The disease proved very harmful. While sound plants produced on the average 5.54 g seeds, the diseased ones yielded only 0.96 g. The average diameter of a sound bulb (in the case of Zittau onion) was 5.085 ± 0.12 cm,

while that of a diseased bulb was 2.352±0.059 cm. The diseased bulbs did not reach normal maturity. Instead of being spheroid as are normal bulbs, they remained of an elongated shape, and the greater part of them germinated already in autumn (Fig. 2). Small diseased bulbs have no chance to survive till spring, and possibly this partly explains why the perecentage of diseased plants is much smaller when small bulbs are planted. An interesting consequence of the disease in the offspring (the phenomena of blastophoria) has been observed. Seeds of diseased plants show a lower viability than do seeds of normal onions. In 7 successive series of experiments, in which a total of 10000 seeds had been sown, normal seeds showed a viability of 64%—83% or, on the average, 76.3%, while the diseased ones showed viability of only 42—53% or, on the average, 46.8%. Seedlings obtained from seeds of diseased plants were much weaker than those from seeds of sound plants. The average height of a normal seedling was 5.08±0.1 cm, while of a diseased one of the same age was 2.36±0.04. The root system of seedlings from diseased seeds was not so well developed as that of normal ones. The seedlings derived from normal plants had, on the average, 5 rootlets, while

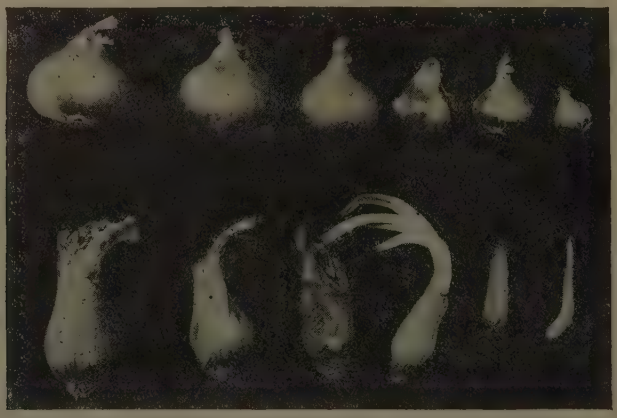


Fig. 2. Bulbs of *Allium Cepa*. Above is shown a plant derived from sound parents; below, one derived from diseased parents.

those derived from seeds of diseased plants had no more than 3 rootlets at the same time.

Using various methods of staining, both intra-vitum and after fixation in Zenker's fluid, we investigated different plant tissues (roots, bulbs, leaves and flowers) in search of the pathogenic agent, but found neither bacteria nor fungi.

In its pathological, anatomical and cytological aspects, the disease is characterized by the following features. In the strongly affected parts of the leaf, the differentiation of the mesophyll is lost, the cells of the palisade parenchyma become indistinguishable from the cells of the deeper layers. The phenomena of the hypoplasia of the stomata are to be observed and the slit between the guard cells may fail to form. Not infrequently, due to additional divisions, three or four guard cells are formed instead of two. A strong vacuolization of the cytoplasm, similar to that described by Dufrenoy (2) in virus diseases in plants, is observed to occur. Near the nucleus dense

accumulations of granulated cytoplasm are to be observed, so that often the outlines of the nucleus are invisible in diseased cells. In strongly affected cells, intracellular inclusions were found, consisting of loose homogeneous bodies, of which one or two, seldom more, were observed to lie near the nucleus. These bodies, similarly to the cytoplasm, are stained with acid stains, such as eosin or acid fuchsin. A differential staining may be effected by using Giemsa's stain in formaline, the cytoplasm taking on a dark blue colour and the *X*-bodies becoming red. Under the influence of the disease, the size of the plastids in the diseased areas decreases, this decrease being the stronger the greater is the loss of chlorophyll. When subjected to intravitum staining with neutral-red, the epidermis of the leaf turns reddish-lilac above the green areas, and yellowish-orange above strongly diseased areas.

By inoculating plants with the juice of diseased individuals by means of a needle, only a small percentage of the experimental plants were infected. Later on, it was discovered that artificial infection is very easy to produce by rubbing the leaves with emery paper wetted in the juice of diseased plants. In an experiment made with bulbs placed out of the reach of insects, of a total of 33 bulbs 29 were infected, while out of the 72 control plants only 1 caught the disease. In another experiment, where the plants were not isolated from insects, out of a total of 50 plants, 34 caught the disease, while, of the controls, only 8 were infected.

In experiments with the inoculation of the leaves, the incubation period lasted from 10 to 14 days.

These experiments prove the infectious nature of the disease. The latter posesses all the characteristics of a virus disease: infectiousness in the absence of visible parasites, the fact that chlorophyll grains are affected, presence of intracellular inclusions. Of course, in order to prove that the disease in question is really a virus disease, it is necessary to filtrate the pathogenic agent through a bacterial filter.

We know from the literature that there are two virus diseases of the onion: the yellow dwarf ([3]) and the mosaic disease. The former has been described in detail [Melhus Reddy, Henderson and Vestal ([4])], while the latter has only been briefly dealt with by Atanasoff ([1]). Neither of these diseases affect the flowers. Melhus, Reddy, Henderson and Vestal state expressly that the flowers undergo no changes during the disease. This justifies us in assuming that the disease in question is new, not only for the Soviet Union, but in general.

Since it has been found possible to cure some of the virus deseases by heating, as was the case for instance with sereh of the sugar cane and the yellow in peaches, we tried to cure the bulbs by heating them. Our attempts failed. A temperature of 55—58°, to which our bulbs were exposed for 2 hours, did not cure the plants. When kept at this temperature for 3 hours, the bulbs perished. They perished likewise when kept at 65° for 2 hours.

The disease under consideration demands further study, and, first of all, it is necessary to ascertain the way it propagates and its distribution over the territory of the Soviet Union.

We wish to acknowledge our indebtedness to the staff of the Agricultural Experimental Station at Alexeyevka for the assistance extended to us in the course of the present work.

Institute of Microbiology.
Academy of Sciences of the USSR.
Moscow.

Received
27. IV. 1937.

REFERENCES

[1] Atanasoff, Bull. de la Soc. Botan. de Bulgaria, II (1928). [2] Dufrenoy, Phytopath ZS., 5 (1932). [3] W. Henderson, Phytopath., 22 (1932). [4] Melhus, Reddy, Henderson, Vestal, Phytopath., 19 (1929). [5] Рыжков, Вирусные болезни растений. Общая и частная вирология, Москва (1935).

PRIX D'ABONNEMENT

Un an — 36 livraisons . . . . . $ 15
Six mois — 18 livraisons . . . $ 7.50
(Séparément $ 0.50 le numéro)
«Nouvelle Série» pour 1933 — 6 livraisons . . . $ 2
«Nouvelle Série» pour 1934 — l'année complète épuisée; volume III (9 livraisons) et volume IV (9 livraisons) en vente à $ 3 le volume.

URSS — Mezhdunarodnaja Kniga, 18, Kuznetskij Most, Moscou (Compte courant n° 263 Banque d'Etat de l'URSS, 12, Neglinnaja, Moscou).
NORTH and SOUTH AMERICA — BOOKNIGA CORPORATION. 255 Fifth Avenue. New York, N. Y.
GREAT BRITAIN — W. H. SMITH & son, Ltd. Strand House, Portugal St London, W. C. 2.
FRANCE — MESSAGERIES HACHETTE. Service Abonnements. 111, Rue Réaumur. Paris, 2e.
SUISSE — Administration du Journal „LE TRAVAIL". 39, Rue de Lausanne. Genève.
ROUMANIE — Deposit de Ziare Luvru. Calea Victoriei 30. Bucarest.
Ç. S. R. — MELANTRICH, Akc. Spol. Knihkupectvi — odd. slovanskych knih. Vaclavské nam 42. Praha II. (Ucet post. spor. 20. 208).
BELGIQUE — LIBRAIRIE A.U.S. 38. Vieille Halle-aux-Blés. Bruxelles (C. C. P. 1861. 15 P. Driessen).
CHINA — AMERICAN BOOK & SUPPLY Co 841 Bubbling Well Road. P. O. Box 1798. Shanghai.
HOLLAND — J. GINSBERG. University Bookseller. Kort Rapenburg, 17. Leiden
LIETUVA — „MOKSLAS". Knygynas. Laisves Aleja, 46. Kaunas.
TURQUIE — LIBRAIRIE HACHETTE. Succursale de Turquie. 469, Avenue de l'Indépendance. Boite Postale Pera 2219. Istanbul.
POLSKA — GEBETHNER & WOLFF. Filja I. Krakowskie Przedm. 15. Warszawa.
ROSSICA. Chmielna 5. Warszawa.
KSIEGARNIA GIRSZOWSKIEGO. Zamkowa 13. Wilno.
PALESTINE — SAFRAN Co. Jaffa Rd. 36. P. O. B. 223. Haifa.
BULGARIA — T. F. TSCHIPEFF. Buchhandlung. Boulevard Dondoukoff 20/22 Sofia. (Postscheckkonto-Sofia 540).
SVERIGE — C. E. FRITZE'S Kgl, Hofbuchhandlung. Fredsgatan, 2 Stockholm.
LATVIJA — A./S. VALTERS un RAPA. Teatra iela Nr. 11. Riga (Pasta tek. rek Nr. 675)
NORGE — JOHAN GRUNDT TANUM. Tidligere Aschehougs Boghandel. Karl Johans gt. 43. Oslo.

Les «Comptes Rendus», largement expédiés à l'étranger dans les centres scientifiques les plus importants et dans les bureaux de nombreuses revues, sont le seul organe scientifique en URSS, donnant tous les dix jours un compte rendu des plus récentes données de la science soviétique.

Prix — $ 0.50